Fourier Analysis on Number Fields and Hecke's Zeta Functions - Tate's Thesis

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1 Abstract

In the early 20th century, Hecke attempted to find a further generalization of the Dirichlet L-series and the Dedekind zeta function. In 1920, he [14] introduced the notion of a Grössencharakter, an ideal class character of a number field, and established the analytic continuation and functional equation of its associated L-series, the Hecke L-series. In 1950, John Tate [27], following the suggestion of his advisor, Emil Artin, rewrote Hecke's work. Tate provided a more elegant proof of the functional equation of the Hecke L-series by using Fourier analysis on the adeles and employing a reformulation of the Grössencharakter in terms of a character on the ideles. Tate's work now is generally understood as the GL(1) case of automorphic forms [2] and in an extension to Langlands Program.

2 Introduction

We introduce our paper with a well-known function, Riemann zeta function, which will play an important role in our development of the thesis. It is defined as an absolutely convergent series

$$\zeta(s) = \sum_{n=1} \frac{1}{n^s}$$

for complex numbers s such that $R(s) > 1$. If we let $s = 1$, then series diverges and becomes a harmonic series.

In 1859, Riemann[25] showed that the function $\zeta(s)$ is analytically continuous on a complex plane to a holomorphic function at $s \neq 1$. Thus, the residue of a pole at $s = 1$ is 1. He also proved the following equation

$$\xi(s) = \xi(1 - s)$$

where $\xi(s) = \pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s)$ and

$$\Gamma(s) = \int_0^\infty e^{-t}t^{s-1}dt$$

It can be shown that $\Gamma(s)$ is convergent for $R(s) > 1$ and can be meromorphically continued to the s plane with poles at negative integers. Riemann also showed that $\zeta(s)$ has infinitely many zeros on the critical strip of $0 < R(s) < 1$ and claimed that all zeros lie on the line $R(s) = 1/2$. This is known as Riemann Hypothesis and it has not been proved till date.

Riemann also proved the prime number theorem which states that

$$\lim_{x\to\infty} \frac{(\frac{x}{logx})}{\pi(x)} = 1$$

where $\pi(x)$ is the prime number function for $x \in \mathbb{R}$ such that $p \leq x$ for p prime numbers.

The Generalised Riemann Hypothesis, which was proven using Dirichlet L-series, states that for every Dirichlet character χ, the zeros of $L(\chi, s) = 0$ in the critical strip lies on the line $R(s) = 1/2$. We will learn more about L-functions and Dirichlet characters in the upcoming sections, as they will play an important role in our understanding of Hecke's zeta function.

Algebraic number theory is the study of Diophantine equations $x^n + y^n = z^n$. This led mathematicians to study unique factorization of number fields. With these results, mathematicians were almost able to prove Fermat's Last Theorem. This led to introduce the concept of ideals and Kummer was the first to study ideals on number fields[10]. If the ring of integers of a number field is a principal ideal domain (i.e. every ideal is generated by a single element), then the ring of integers is a unique factorization domain. So,the failure of the ring of integers of a number field to be a principal ideal domain that prevents the ring of integers from being a unique factorization domain. The fractional ideals of a number field form a group under multiplication. The ideal class group of the number field is the quotient group of all fractional ideals by principal ideals, and thus is an object that measures how unique factorization fails. Investigating the prime ideals of a number ring can help us to understand the ideal class group , and hence the failure of unique factorization.

We know that the Riemann zeta function gives information about the distribution of rational primes, Dedekind's work generated the Dedekind zeta function [7] for a number field K hoping to gain a better understanding of the primes in a number ring, and thus make progress on solving Fermat's Last Theorem. More information about the values of integer points of $\zeta(s)$ can be studied in Algebraic K-Theory.

As Dirichlet generalised the Riemann-zeta function, similarly, Hecke wanted to generalise the Dedekind zeta function to an L-function of a character on a number field. He created a special multiplicative function on the ideals of o_K, called Grossencharackter.

Hecke [14] proved that the function $L(s, \chi)$ has meromorphic continuation to the whole s-plane and satisfies a functional equation. However, he was not able to explicitly describe the factors that arise in the functional equation. In 1940, Chevalley [3] introduced the notion of the idele-class group. In 1950, John Tate [27] used the idele-class group to redefine notion of a Grössencharakter; his definition eliminated many of the difficulties associated with constructing a Grössencharakter.

The following paper attempts to understand the machinery of Tate's thesis and how seemingly disparate fields of mathematics connect to one another. Tate's work both inspired and led to the study of automorphic forms and representations and, more generally, to the Langland's Program, itself one of the most overarching theories in mathematics and number theory. In the following chapters, we will gather some background knowledge which is necessary, beginning with topological groups, Haar measure, Pontryagin duality and Fourier Inversion formula in next section 3. In section 4, we will briefly study the results from local and global fields. This section contains a short summary of how the existence and uniqueness of the Haar measure, explicitly the module of automorphism, can be used to classify locally compact fields, followed by the next section, where we introduce the restricted direct product and its topology; results about the quasi-characters, characters, the dual group, the Haar measure of the restricted product and the results of adeles and ideles are also proved.

Finally, we introduce the main content of Tate's thesis and explore in detail the Schwartz-Bruhat functions, the Poisson summation formula and its extensions, the Riemann-Roch Theorem, the proof of the meromorphic continuation and functional equation of the Hecke L-function attached to an idele-class quasi character and the volume of the norm-one idele-class group and thereby provide the residues of the Hecke and Dedekind zeta function at $s = 1$.

3 Topological Groups, the Haar Measure, and Pontryagin Duality

In this section, we will primarily follow the chapters from Ramakrishnan and Valenza's Fourier Analysis on Number Fields [24] and Folland's book, Real Analysis: Modern Techniques and Their Applications. Most of the short proofs will be given however the reader will be referred to the text for others.

3.1 Topological Groups

Definition 3.1.1 A topological group is a group G with a topology such that $(g, h) \mapsto gh$ from $G \times G$ to G and $g \mapsto g^{-1}$ is continuous.

Proposition 3.1.2 *A group G is a topological group if and only if for all $g.h \in G$ and any neighbourhood W of gh^{-1}, there exists an open neighbourhoods of U of g and V of h such that $UV^{-1} \subseteq W$*

Proof: Let G be a topological group and let W be a neighbourhood of gh^{-1}. Let us take W to be open and by definition of neighbourhood, we can there exists an open set W_1 that contains gh^{-1} and in W. We can show the following multiplication map $f : G \times G \to G$ defined by $(g, h) \mapsto gh$. As $f^{-1}(W)$ and constitutes basis for $G \times G$, then there exists open sets U and V_1 of G containing g and h^{-1}, respectively, such that $U \times V_1 \subseteq f^{-1}(W)$. As inversion is continuous, V_1^{-1} is open set containing h. Thus, we define $V_1^{-1} = V$ and $V^{-1} = V_1$ is an open neighbourhood of h^{-1}. Thus, there exists an open neighbourhoods of U of g and V of h such that $UV^{-1} \subseteq W$.

□

By using discrete topology on G, it will be a topological group. It is also a translation invariant, where by fixing an element, we consider its right or left translation, which is a homeomorphism from $G \mapsto G$. Let $g \in G$ and $U \subseteq G$, the following are equivalent:

(i) U is open

(ii) gU is open

(iii) Ug is open

Definition 3.1.3 Let X be a topological space and let S be a subset of $Homeo(X)$, the set of all homeomorphism from X to itself. Then X is said to be a homeogenous space under S if $\forall x, y \in X$, there exists $f \in S$ such that $f(x) = y$.

Proposition 3.1.4 *Every topological group is translation invariant and homogenous under itself ($S = G$). Furthermore, local neighbourhood base at the identity determines a local base at all $g \in G$*

Proof: Refer to the text by Ramakrishnan and Valenza

□

Examples :

(i) Any group G imposed with the discrete topology is a topological group.

(ii) Any group G imposed with the trivial topology is a topological group.

(iii) Every subgroup of a topological group, imposed with the subspace topology, is a topological group.

(iv) The groups ($\mathbb{Z}$, +), ($\mathbb{Q}$, +), with the subspace topology induced by the Euclidean topology on $\mathbb{R}$ are topological groups.

Proposition 3.1.5 Let G be a group and assume the topology on G is induced from metric, d. Then G is a topological group if and only if the following condition hold:

(i) For all $\epsilon > 0$ and $g_1 g_2 \in G$ there exists a $\delta > 0$ such that $d(g_1 g_2, h_1 h_2) < \epsilon$ whenever $d(g_1, h_1) < \delta$ and $d(g_1, h_2) < \delta$

(ii) For all > 0 and $g \in G$ there exists $\delta > 0$ such that $d(g^{-1}h^{-1}) < \epsilon$ whenever $d(g, h) < \delta$

Proof: Proof is trivial

□

Proposition 3.1.6 Let G_1 and G_2 be topological groups. The direct product $G_1 \times G_2$ imposed with product topology and group operation is a topological group

Proof: Since G_1 and G_2 are topological groups, we can say that there exists open sets U_1, V_1, U_2, V_2 of g_1, h_1, g_2, h_2 respectively such that $U_1 V_1^{-1} \subseteq W_1$ and $U_2 V_2^{-1} \subseteq W_2$, where W_1 and W_2 are contained in neighbourhood W. Thus, $U_1 \times U_2$ is a neighbourhood of (g_1, g_2) and $V_1 \times V_2$ is a neighbourhood of (h_1, h_2) such that $(U_1 \times U_2)(V_1 \times V_2)^{-1} \subseteq (W_1 \times W_2) \subseteq W$. Therefore, $G_1 \times G_2$ is a topological group.

□

Let G be a topological group, then the subset S is called symmetric if and only if $S = S^{-1}$.

Proposition 3.1.7 Let G be a topological group, then the following assertions hold:

(i) Every neighbourhood U of the identity contains a neighbourhood V of the identity such that $VV \subseteq U$.

(ii) Every neighborhood U of the identity contains a symmetric neighborhood V of the identity.

(iii) If H is a subgroup of G, so is its closure.

(iv) Every open subgroup of G is also closed.

(v) If K_1 and K_2 are compact subsets of G, so is K_1K_2.

Proof : Proof is trivial

□

Let f be a continuous from a group G to $\mathbb{R}$ or $\mathbb{C}$, we define that f is left uniformly continuous if, for all $\epsilon > 0$, there exists a neighbourhood V of identity such that

$$\|L_h f - f\|_v < \epsilon; \forall h \in V$$

where $L_h f(g) = f(h^{-1}g)$ as left translate and $R_h f(g) = f(gh)$ as right translate

Let $C_c(G)$ be the space of continuous functions on G with compact support

Proposition 3.1.8 *Let G be a topological group. Every function $f \in C_c(G)$ is both left and right uniformly continuous.*

Proof: Let $K = sup(f)$ and $\epsilon > 0$. As f is continuous, we can say that U_g is an open neighbourhood $\forall g \in K$ and

$$|f(gh) - f(h)| < \epsilon$$

From the previous proposition, we know that there exists a neighbourhood V_g, of the identity such that $V_gV_g \subseteq U_g$. Consider the cover gV_g of K, which we can reduce to a finite subcover,$g_iV_{g_i}$ where $i = 1, 2, ...$ by compactness. Let $V = \bigcap_{i=1}^{n} V_g$. Let $h \in V$ and $g \in K$. If $g \in K$, then there exists an $i \in (1, 2, ...n)$ such that $g \in g_iV_{g_i}$. Using the triangle inequality, we get

$$|f(gh)f(g)| \leq |f(gh)f(g_i)| + |f(g_i)f(g)|$$

Both terms on the right are bounded by because $g_i^{-1}g = g_j^{-1}(gh)h^{-1} \in V_{g_j}V_{g_i} \subseteq U_{g_j}$. Thus, f is right uniformly continuous in K. If g is not in K, then we need to bound $f(gh)$. If $f(gh) \neq 0$, then $gh \in supp(f)$, and hence $gh \in g_jV_j$ for some $j1, ..., n$. Therefore, $f(gh)f(g_j) <$. Also $g_{ij}^{-1}g = g_j^{-1}(gh)h^{-1} \in V_{g_j}V_{g_j} \subseteq U_{g_j}$, so $|f(g_j)| < \epsilon$. Finally, $|f(gh) - f(h)| < 2\epsilon$

□

Proposition 3.1.9 *Let G be a topological group. Then the following assertions are equivalent:*

(i) G if T_1

(ii) G is Hausdorff

(iii) The identity e is closed in G

(iv) Every point of G is closed in G

Proof: Proof is trivial

□

Let G be a topological group and H be its subgroup. Then we can say that, G/H is a set of left cosets with quotient topology such that $\rho : g \mapsto gH$ is continuous. U is open in G/H is a group if and only if $\rho^{-1}(U)$ is open in G. Under coset multiplication, G/H is a group if and only if H is a normal subgroup of G. Thus, G/H can be said as a topological group.

Proposition 3.1.10 *Let G be a topological group and let H be a subgroup of G. Then the following assertions hold:*

(i) The quotient space G/H is homogeneous under G

(ii) The canonical projection $\rho : G \to G/H$ is an open map

(iii) The quotient space G/H is T_1 if and only if H is closed

(iv) The quotient space G/H is discrete if and only if H is open. Moreover, if G is compact, then H is open if and only if G/H is finite

(v) If H is normal in G, then G/H is a topological group with respect to coset multiplication and the quotient topology

(vi) Let H be the closure of e in G. Then H is normal in G, and the quotient group G/H is Hausdorff with respect to the quotient topology

Proposition 3.1.11 *Let G be a Hausdorff topological group. Then:*

(i) The product of a closed subset F and a compact subset K is closed

(ii) If H is a compact subgroup of G, then $\rho : G \to G/H$ is a closed map

Proof: (i) Let $z \in \overline{FK}$. So, z is the limit of convergent net $f_j k_{j_{j\in I}} \subset FK$, where $f_{j_{j\in I}} \in F$ and $k_{j_{j\in I}} \in K$. Since K is compact, there exists a convergent subnet $\kappa_j \in K$ that converges to a point $k \in K$. Note that since $f_j k_j$ converges, then we can replace $f_j k_j$ with $f_j \kappa_j$. Consider U an open neighborhood of e in G. As shown above, there exists an open neighborhood V of e such that $VV \subseteq U$. The nets $z^{-1} f_j \kappa_j$ and $\kappa_j^{-1} k_k$ both converge to e and thus lie in V. Since $VV \subseteq U$, then the product of the nets,$z^{-1} f_j k$, eventually lie in U. Consequently, $lim f_j = zk^{-1}$ and $z = zk^{-1}k \in FK$

(ii) Let C be closed in G. Then we must show that $\rho(C)$ is closed in G/H. However, under the quotient topology, this reduces to showing that $\rho^{-1}\rho(C) = CH$ is closed in G. By part (i), CH is closed in G since H is compact and C is closed.

□

3.2 Locally Compact Fields

Definition 3.2.1 A ring R with operations "+" and "." such that $(R, +)$ is a topological group and such that $M : R \times R \to \mathbb{R}$ defined by $(r, s) \mapsto r.s$ is continuous is called a topological ring. Similarly, we can define topological field.

Definition 3.2.2 A topological space is locally compact if every point of the space admits a compact neighborhood. A topological group G that is both locally compact and Hausdorff is called a locally compact group. A topological field F that is both locally compact and Hausdorff is called a locally compact field.

Proposition 3.2.3 *Any locally compact subset of a Hausdorff space is the set theoretic difference of two closed sets or, equivalently, is the intersection of an open and closed set.Consequently, any locally compact dense subset of a Hausdorff space is open.*

Proof: Let S be a compact subset of a Hausdorff space X. We can find an open neighborhood U in S of $s \in S$ such that $Cl_s U$ is compact in S. Since U is open in S, then there exists V, open in X, such that $U = V \cap S$. Then $Cl_x(V \cap S) \cap S = Cl_x U \cap S = Cl_s U$ is compact. So, $Cl_x(S \cap V) \cap S$ is closed in X and contains $S \cap V$, thus $Cl_x(S \cap V) \subset S$. Hence, $Cl_x S \cap V$ is a neighbourhood of s in $Cl_x S$, which is contained in S. Therefore, S is open on $Cl_x S$. Any open set in $Cl_x(S)$ form $B \cap Cl_x(S)$ where B is open in X. If S is dense and locally compact, then, as shown, $S = O \cap C$, where O and C are, respectively, open and closed in X. Since $S = A - T$ where A, T are closed in X, then pick $x \in A^c$, which is open. Let U be open neighbourhood of x in A^c, which is open. This contradicts with density of S. Therefore, $A^c = \emptyset$ and $A = X$, which implies $S = X - T$. Consequently, S is open.

□

Proposition 3.2.4 *Let $(G_i)_i \in I$ be a set of locally compact groups such that G_i is compact for all but finitely many $i \in I$. Then*

$$\prod_{i=1} G_i$$

is locally compact

Proof: Let $S = i \in I : G_i$ not compact. By hypothesis, this set is finite. By Tychonoff's theorem, the possibly infinite product

$$\prod_{nu \notin S} G_i$$

is compact. Furthermore, since G_i, $i \in S$ is locally compact, then the finite product $\prod_{i \in S} G_i$ is locally compact. Indeed, let $(g_i)_{i \in S}$ be a point in $\prod_{i \in S} G_i$. Since, G_i is locally compact, then for all $i \in S$, there exists a locally compact neighbourhood $K_i \subset G_i$ of g_i. Let $K = \prod_{i \in S} K_i$. Then K is a compact neighbourhood of $(x_i)_{i \in S}$ in the direct product $\prod_{i \in S} G_i$ since a product of finitely many compact sets is compact. Thus, the full product is locally compact.

□

Proposition 3.2.5 *If G is locally compact group and H is a closed subgroup, then G/H is a locally compact group.*

Proof: Proof is trivial using the Propositions proved previously.

□

3.3 Haar Measure

In this section, we will recall some important results of Measure theory and observe how they will be useful in developing our thesis.

Recall that the Borel σ-algebra for a topological space X is the smallest σ-algebra containing all open sets. A positive measure μ on a measure space (X, M) is a function $\mu : M \to \mathbb{R}_+ \cup (\infty)$ that is countably additive, where $\mathbb{R}_+$ is set of nonnegative reals.

$$\mu\Big(\bigcup_{n=1}^{\infty} A_n\Big) = \sum_{n=1}^{\infty} \mu(A_n)$$

where A_n is a collection of disjoint sets in M. Let μ be a Borel measure on X, a locally compact Hausdorff space and let E be a Borel subset of X. We can say that μ is outer regular on E if

$$\mu(E) = inf\{\mu(U) : E \subseteq U, U is open\}$$

and that μ is inner regular on E if

$$\mu(E) = sup\{\mu(K) : K \subseteq E, K is compact\}$$

A measure μ is said to be regular if every Borel set in X is both inner and outer regular.

Definition 3.3.1 A Radon measure on a X, a locally compact Hausdorff space, is a Borel measure that is finite on compact sets, outer regular on all Borel sets, and inner regular on all open sets.

A measure μ is said to be left translation invariant, if for all Borel measure sets E in G

$$\mu(s.E) = \mu(E)$$

for all $s \in G$, where G is locally compact topological group and μ is Borel measure on G.

Definition 3.3.2 A left Haar measure (respectively, right Haar measure) on a locally compact group G is a nonzero Radon measure μ, which is left translation invariant (respectively, right traslation invariant). A bi-invariant Haar measure on a locally compact group G is a nonzero Radon measure that is both right and left invariant.

Proposition 3.3.3 *Let G be a locally compact group with nonzero Radon measure μ. Then the following assertions are true:*

(i) The measure is a left Haar measure on G if and only if the measure $\tilde{\mu}$, defined by $\tilde{\mu}(E) = \mu(\tilde{E} - 1)$, is a right Haar measure on G

(ii) The measure μ is a left Haar measure on G if and only if

$$\int_G L_s f d\mu = \int_g f d\mu$$

(iii) If μ is a left Haar measure on G, then μ is positive on all nonempty open subsets of G

$$\int_g f d\mu > 0$$

(iv) If μ is a left Haar measure on G, then $\mu(G)$ is finite if and only if G is compact

Proof: (i) As the inversion is a homeomorphism, then E^{-1} is Borel if and only if E is Borel. Then we have that $\tilde{\mu}(Es) = \tilde{\mu}\ (E)$ for all $s \in G$ and for all Borel sets in E if and only if $\mu(s^{-1}E^{-1}) = \mu(E^{-1})$ for all $s \in G$ and Borel sets E

(ii) As μ is Haar measure on G, then the equality of integrals follows by definition of simple functions, as they are the finite linear combinations of characteristic functions on G. Conversely, we can also use by the Riesz representation theorem, explicitly recover the Radon measure μ of any open subset $U \subseteq G$ as follows:

$$\mu(U) = sup\{\int_G f d\mu : f \in C_c(G), \|f\|_u \leq 1\}$$

From this one sees at once that if left integral is left translation invariant, then $\mu(sU) = \mu(U)$ for all open subsets U of G. The result now extends to all Borel subsets of G because a Radon measure is by definition outer regular.

(iii) As $\mu > 0$ and its left Haar measure, so we can say that $\mu(s_i U) = \mu(U)$. Let U be an open set in G. Let $K \subseteq G$, then K is compact, then there exists $s_1, s_2, s_3....s_n$ in G such that $K \subseteq \bigcup_{i=1}^{n}(s_i U)$. With the cover of finitely many translates on compact set, we can prove existence of the Haar measure on a locally compact group. Using the result from left Haar measure, if $f \in C_c$, then there exists a compact set $K^{'}$ such that $f > 0$. Furthermore, there exits a set $U^{'} \subseteq K$ with $\mu(U) > 0$ such that $f > R$ for some constant $R > 0$. Then

$$\int_G f d\mu = R\mu(U) > 0$$

□

Theorem 3.3.4 *Every locally compact group G admits a left (or right) Haar measure. Furthermore, this measure is unique up to multiplication by a positive real constant.*

Proof: Proof involves using the Riesz Representation theorem mentioned below and can be found in full extent in Rudin [26]

□

We will state below the Riesz Representation Theorem, as taken from Rudin [26], Chapter 2; it is the essential ingredient in the proof of the existence of a Haar measure for locally compact groups.

Theorem 3.3.5 *Let X be a locally compact Hausdorff space and let Lambda be a positive linear functional on $C_c(X)$. Then there exists σ-algebra M in X which contains all Borel sets in X, and there exists a unique positive measure μ on M, which represents |Lambda in the sense that :*

(i) $\Lambda f = \int f d\mu \forall f \in C_c(X)$ satisfies the following properties

(ii) $\mu(K) < \infty$ for all compact sets $K \subset X$

(iii) μ is outer regular on $E \in M$

(iv) μ is inner regular on all open sets and all $E \in M$ such that $\mu(E) < \infty$

(v) If $E \in M$, $A \subset E$ and $u(E) = 0$, then $A \in M$. We say a measure is complete if it satisfies this property.

Corollary 3.3.6 *Let μ and M be as above*

(i) μ is a Radon measure

(ii) Every σ-compact set has σ-finite measure

(iii) If $E \in M$ and E has σ-finite measure, then E is inner regular

(iv) If X is σ-compact, then μ is regular

(v) If μ is σ-finite, then σ is regular

3.4 Pontryagin Duality and the Fourier Inversion Theorem

In this section, we will discuss continuous characters of G. Its a topological group of continuous homomorphisms from a group G to S^1. The group of continuous characters form a group under multiplication and this group is called the Pontryagin dual of G and is denoted by $\hat{G}$. The Pontryagin duality states that G and $\hat{G}$ are mutually dual.

Let us now focus on the abelian topological group G. We will write $\hat{G}$, as the multiplicative group of continuous complex characters of G. That is $\hat{G} = Hom(G, S^1)$. Let us now take a look at the Proposition below about abelian groups

Proposition 3.4.1

(i) $\hat{\mathbb{R}} \cong \mathbb{R}$ *with pairing* $< x, \xi \geq e^{2\pi i \xi x}$

(ii) $\hat{S^1} \cong \mathbb{Z}$ *with pairing* $< \alpha, \alpha \geq \alpha^n$

(iii) $\hat{\mathbb{Z}} \cong S^1$ *with pairing* $< n, \alpha \geq \alpha^n$

(iv) $\widehat{\mathbb{Z}/n\mathbb{Z}} \cong \mathbb{Z}/n\mathbb{Z}$ *with pairing* $< m, k \geq e^{2\pi i \frac{mk}{n}}$

In Pontryagin dual field, $\mathbb{R}$ *is the dual group of additive group of the field* $(\mathbb{R}, +)$

The dual group of the locally compact field $\mathbb{R}$ is itself. We will see in the upcoming sections that locally compact non-discrete fields (local fields) are self-dual.

Proposition 3.4.2 *The Pontryagin dual of* $G_1 \times G_2$ *is isomorphic to* $\hat{G}_1 \times \hat{G}_2$

Let G be a group and X a subset of G. We define $X^n \subseteq G$ where $n \in \mathbb{N}$

$$X^{(n)} = \{\prod_{j=1}^{n} x_j : x_j \in X, j = 1, ..., n\}$$

We will now impose $\hat{G}$ with compact-open topology. We define a neighbourhood base for trivial characters in $\hat{G}$.

$$W(K, V) = \{\chi \in \hat{G} : \chi(K) \subseteq V\}$$

where K is a compact subset of G and V is a neighborhood of the identity in S^1.

Thus, if G is compact, the compact-open topology coincides with topology of uniform convergence and if G is a separable locally compact abelian group, then $\hat{G}$ is metrizable.

The following lemma will present me the important properties of compact-open topology on $\hat{G}$

Lemma 3.4.3 *Let m be a positive integer and suppose that $x \in \mathbb{C}$ such that $x, x^2, x^3, ...x^m$ lie in $N(1)$. Then $x \in N(1/m)$. Consequently, if U is a subset of G containing the identity and $\chi : G \to S^1$ is a group homomorphism such that $\chi^{U^m} \subseteq N(1)$, then $\chi(U) \subseteq N(1/m)$*

Proof: We know that there is a universal covering space of S^1 is $\mathbb{R}$ with continuous surjective map

$$\phi : \mathbb{R} \to S^1$$

The kernel of ϕ is $\mathbb{Z}$. For $\epsilon \in \mathbb{R}$ such that $0 < \epsilon \leq 1$, define $N(\epsilon) \subseteq S^1$ as

$$N(\epsilon) = \phi((-\epsilon/3, \epsilon/3))$$

We can prove this lemma by induction. For $m = 1$, it is trivial. Let r be a positive integer such that $x^r \in N(1)$. Then $x \in (-1/3, 1/3)$ which implies that there exists a $y \in N(1/r)$ such that $x^r = y^r$. Hence, the quotient of x/y is an r^{th} root of unity. As $\phi(q/r) = e^{2\pi iq/r}$ is an r^{th} root of unity for all $q \in \mathbb{Z}$ then $x \in N(1/r)\phi(q/r)$. We claim that for all $r > 0$

$$N\left(\tfrac{1}{r}\right) \cap N\left(\tfrac{1}{r+1}\right)\phi\left(\tfrac{q}{r+1}\right) \neq \emptyset \implies q = 0$$

By definition we know that

$$N\left(\tfrac{1}{r+1}\right)\phi\left(\tfrac{q}{r+1}\right) = \{e^{2\pi it/3} : t \in \left(\tfrac{3q-1}{r+1}, \tfrac{3q+1}{r+1}\right)\}$$

and

$$N\left(\tfrac{1}{r}\right) = \{e^{2\pi it/3} : t \in \left(\tfrac{-1}{r}, \tfrac{1}{r}\right)\}$$

The above sets will not have intersection unless

$$\tfrac{1}{r} > \tfrac{3q-1}{r+1} \implies 2r + 1 > 3qr$$

which cannot hold unless $q = 0$

Let us suppose that $x \in N(1/r)$ and $x^{r+1} \in N(1)$. Then using the previous argument again, we obtain $x \in N(1/r+1)\phi(q/r+1)$ where $0 \leq q < r+1$. Then $x \in N(1/r) \cap N(1/(r+1))\phi(q/(r+1))$ which implies $q = 0$, and hence $x \in N(1/(r+1))$. Consequently, it follows by induction that if $x^1, x^2, ..., x^m$ lie in $N(1)$, then $x \in N(1/m)$.

Similarly, let $g \in U \subseteq G; e \subseteq U$. Then $\{g^1, g^2, ...g^m\} \in U^{(m)}$. Therefore, if $\chi(U^{(m)}) \subseteq N(1)$, then $\chi(g_1), \chi(g^2)...\chi(g^m) \in N(1)$. Thus, $\chi(U) \subseteq N(1/m)$.

□

In the following proposition, we will observe that the dual group of a locally compact group is locally compact. Furthermore, we will see that the dual group of a compact group is discrete and that the dual group of a discrete group is compact. These three facts will be essential in proving that a local field is isomorphic to its dual.

Proposition 3.4.4

(i) A group homomorphism $\chi : G \to S^1$ is continuous and hence a character of G, if and only if $\chi^{-1}(N(1))$ is a neighbourhood of the identity in G.

(ii) The family $\{W(K, N(1))\}_K$, indexed over all compact subsets of G, is a neighborhood base of the trivial character for the compact-open topology of $\hat{G}$.

(iii) If G is discrete, then $\hat{G}$ is compact.

(iv) If G is compact, then $\hat{G}$ is discrete.

(v) If G is locally compact, then $\hat{G}$ is locally compact.

Proof: Refer book by Ramakrishnan and Valenza

□

Let G be a locally compact group and let dy be the Haar measure on G. We can say that function $f : G \to \mathbb{C}$ is absolutely integrable if

$$\|f\|_i := \int |f(y)|dy < \infty$$

With respect to function addition, the space of absolutely integrable functions forms a complex vector space. In fact, $\|.\|_1$ is a semi-norm of this vector space. We identify functions $f, g : G \to \mathbb{C}$ if $\|fg\|_1 = 0$ and denote the vector space by $L_1(G)$. Let $f \in L_1(G)$. Then we define $f : \hat{G} \to \mathbb{C}$, the Fourier transform of f, to be

$$\hat{f}(\chi) = \int_G f(y)\overline{\chi}(y)dy$$

For $\chi \in \hat{G}$

Examples 3.4.5

(i) For $G = \mathbb{R}$, we know that $\hat{\mathbb{R}} \cong \mathbb{R}$ and hence we can identify each $t \in \mathbb{R}$ with the character

$$x \mapsto e^{2\pi ixt}$$

Let dx be the Lebesgue measure on $\mathbb{R}$. Let $f \in L^1(\mathbb{R})$.Thus, in this case the Fourier transform reduces to

$$\hat{f}(t) = \int_{\mathbb{R}} f(x)e^{-2\pi ixt}dx$$

Although we are used to thinking of the Fourier transform as a function on $\mathbb{R}$, it is actually function on $\hat{\mathbb{R}}$. The $'t'$ is representing the characters from $\mathbb{R} \to S^1$ given by $\chi(x) = e^{-2\pi ixt}$.

Theorem 3.4.6 *(Fourier Inversion Theorem) There exists a Haar measure $d\chi$ on $\hat{G}$ such that for all $f \in B(G)$*

$$f(y) = \int_{\hat{G}} \hat{f}(\chi)\chi(y)d\chi$$

Note that $\hat{\hat{f}}(y) = f(-y)$. In addition, the Fourier transform $f \mapsto \hat{f}$ identifies $B(G)$ with $B(\hat{G})$.

Proof: See Folland [11], Chapter 4, Theorem 4.32.

□

When defining the Fourier transform on a locally compact group, we must fix a Haar measure. Since the Haar measure is unique up to a positive constant, then any Fourier transform, no matter what measure is fixed when defining it, will only differ from another defined Fourier transform by a constant. Suppose we fix a measure dx on a locally compact group G when defining the Fourier transform. Then the Fourier inversion theorem guarantees such that the Fourier inversion theorem holds for all the existence of a measure $d\chi$ on $\hat{G}$ such that the Fourier transform holds for all $f \in B(G)$. However, if we fix a measure, say $c\dot{d}x$ on G, then the dual measure to this measure is precisely $d\chi/c$.

Theorem 3.4.7*(Pontryagin Duality) The map that associates to $g \in G$ the character $\chi \to \chi(g)$ of $\hat{G}$ is an isomorphism of topological groups G and $\hat{G}$.Hence, G and $\hat{G}$ are mutually dual.*

Proof: Chapter 3, Section 4, of Ramakrishnan and Valenza [24].

□

4 Restricted Direct Topology and the Adeles and Ideles

In this section, we will learn the topology, dual group, and Haar measure of the restricted direct product. Furthermore, we will develop tools for both the integration and the Fourier transform of functions defined on the restricted direct topology. After developing the theory for the general restricted product in the first section, we will explore in the second section how the construction is used in the number theoretic context. We will introduce the additive topological group of adeles of a global field K, denoted A_K and multiplicative topological group of ideles of a global field K, denoted $\mathbb{I}_K$; these will be used extensively in Tate's thesis. The adeles and ideles will help us to do harmonic analysis on global field K.

4.1 Restricted Direct Topology

Definition 4.1.1 Let $J = \{v\}$ be a set of indices for which we are given G_v, a locally compact group, and let J_∞ be a fixed finite subset of J such that for each $v \notin J_\infty$ we are given a compact open subgroup $H_v \leq G_v$. We know that H_v is closed subgroup of G_v. The restricted direct product of G_v with respect to H_v, denoted by G, is defined as

$$\prod_{v \in J} G_v = \{(x_v) : x_v \in G_v\}.$$

And $x_v \in H_v$ for all but finitely many v.

The restricted direct product is a subset of the set-theoretic direct product of the G_v and a subgroup of the group-theoretic product of G_v . The restricted direct product lies somewhere in between the group direct product and the group direct sum (all but finitely many entries are the identity) The topology, which we will call the restricted direct topology on G, is not equivalent to the product topology.

As the restricted direct product is a group with respect to the componentwise group operation, we can specify the neighbourhood base of the identity, to define its topology :

$$B = \{\prod N_v\}$$

where N_v is neighbourhood of $1 \in G_v$ and $N_v = H_v$ for finitely many v.

For any $S \subseteq J$, which necessarily contains J_∞, define G_s by

$$G_s = \prod_{v \in S} G_v \times \prod_{v \notin S} H_v$$

Here it is assumed that $J_\infty \subseteq S$ because we have not necessarily required that there exists H_v for $v \in J_\infty$. G_s is also locally compact.

Proposition 4.1.2 *G_s is an open subgroup of G and the product topology on G_s is identical to subspace topology induced by restricted direct topology defined above.*

Proof: Proof is trivial using the definitions of subspace and product topology $\square$

Corollary 4.1.3 *G is locally compact*

Proof: Since G_s is compact in product topology, which implies that it is compact with restricted direct topology. Furthermore, every $x \in G$ is contained in some G_s for an appropriate set S containing J_∞. So, every element $x \in G$ admits a compact neighbourhood in G_s.

$\square$

Proposition 4.1.4 *A subset Y of G has compact closure if and only if $Y \subseteq \prod K_v$, for some family of compact subsets $K_v \subseteq G_v$, such that $K_v = H_v$ for all but finitely many indices v.*

Proof: Lets assume that Y has compact closure. Let K be the compact closure of Y. As the subsets of the form G_s cover G and thus K, then a finite number of them cover K. Let $S^{'} = \cup_{i=1}^{n} S_i$. Then $G_{S'}$ cover K. Let ρ_v denote the projection of G onto G_v. This projection map will only be continuous in the product topology. Since K is a subset of $G_{S'}$, which has two equivalent topologies– one of which is the product topology, for which ρ_v is continuous–then $\rho_v(K)$ is compact in G_v , and $\rho_v(K) = H_v$ for all but finitely many indices v. Let $K_v = \rho_v(K)$. Therefore, K, and thus Y , is contained in K_v . Now, assume that $Y \subseteq \prod K_v$. Let C be the closure of Y , which is necessarily the smallest closed set containing Y . Since K_v is a closed set containing Y,then $C \subseteq \prod K_v$, which then implies that C is compact.

$\square$

4.2 Restricted Direct Quasi-Characters and Dual Group

Here we will learn about the group of quasi-characters, $Hom_{Cont}(G, \mathbb{C}^\times)$, of the restricted product of G. For $y \in G$, let y_v be the projection onto the factor G_v which maybe identified by closed subgroup of G.

Lemma 4.2.1 *Let $\chi \in Hom_{Cont}(G, \mathbb{C}^\times)$. Then χ is trivial on all but finitely many H_v. Therefore, for any $y \in G$, $\chi(y_v) = 1$ for all but finitely many v, and*

$$\chi(y) = \prod_v \chi(y_v)$$

Proof: Let U be a neighbourhood of 1 in $\mathbb{C}^\times$ that contains no subgroups of $\mathbb{C}^\times$ besides the trivial subgroup. As χ is continuous, then there exists a neighbourhood, V, of identity G, such that $\chi(V) \subseteq U$. We know that open neighborhoods of the identity in the restricted direct topology are of the form of neighbourhood N_v of identity in G_v. Let $V = \prod_v N_v = \prod_{v\in S} N_v \times \prod_{v\notin S} H_v$. Then

$$\chi\big(\prod_{v\notin S} H_v\big) \subseteq U$$

As $\prod_{v\notin S} H_v$ is a subgroup of G and χ is a homomorphism, then $\chi(\prod_{v\notin S} H_v)$ is a subgroup of U. Then

$$\chi\big(\prod_{v\notin S} H_v\big) = \{1\}$$

since the only subgroup of U is the trivial subgroup. Given any $y \in G$, we can factor y into $y_1y_2y_3$, where y_1 is a finite product of the projections of y that lies outside any H_v, and where y_2 is a finite product of the projections of y that lie in some H_v for $v \in S$, and where y_3 is a product of the projections of y, all of which lie in Hv for $v \notin S$. Therefore, χ is trivial on all but finitely many projections of y and $\chi(y) = \prod_v \chi(y_v)$.

□

Lemma 4.2.2 *For each v let $\chi_v \in Hom_{Cont}(G, \mathbb{C}^\times)$ and $\chi_v|H_v = 1$ for all but finitely many indices v. then we have that $\chi = \prod_v \chi_v \in Hom_{Cont}(G, \mathbb{C}^\times)$*

Proof: Let S be a finite set of indices such that $\chi_v|H_v = 1$ for all $v \notin S$. Let m be a cardinality of S. Since $y = (Y_v)$, where $y_v \in H_v$ for all but finitely many v and $\chi_v|H_v = 1$ for all v outside of S. Then $\prod_v \chi_v$ is a well defined quasi character. Let U be a neighbourhood of 1 in $\mathbb{C}^\times$. Then we have a neighbourhood V in $\mathbb{C}^\times$ so that $V^{(m)} \subseteq U$. Since χ_v is a continuous quasi-character of G_v , then for each $v \in S$, there exists a neighborhood N_v of the identity in G_v such that $\chi_v(N_v) \subseteq U$. Then

$$\prod_{v\in S} N_v \times \prod_{v\notin S} H_v$$

is a neighborhood of the identity in G such that

$$\chi\big(\prod_{v\in S} N_v \times \prod_{v\notin S} H_v\big) = \prod_v \chi_v(N_v) \times \prod_v \chi_v(H_v) \subseteq V^{(m)} \subseteq U$$

Thus, χ is continuous.

□

Theorem 4.2.3 *Let G be the restricted direct product of locally compact abelian groups G_v with respect to compact-open subgroups H_v. As topological groups, we have that*

$$\hat{G} = \prod \hat{G}_v$$

where the restricted direct product on the right is taken with respect to subgroups defined by

$$K(G_v, H_v) = \{\chi_v \hat{G}_v : \chi_{v|H_v=1}\}$$

for $v \notin J_\infty$.

Proof: Chapter 4 of Ramakrishnan and Valenza [24].

□

The two lemmas and the theorem mentioned above about the dual group of the restricted direct product will be used in Tate's Thesis. We will use them to construct standard non-trivial adelic structure, ϕ_K of number field K, local field duality and in proving the Poisson summation formula and its useful extension, the Rieman-Roch theorem. The Rieman-Roch theorem is the main result used in proving the meromorphic continuation and functional equation of the global zeta function.

4.3 Restricted Direct Integration and Self-Dual Measure

Let G be the restricted direct product of locally compact groups G_v with respect to compact-open subgroups H_v . Since G is locally compact, then G admits a Haar measure. However,like the characters of G, we want to define a Haar measure on G in terms of Haar measures on G_v .

Proposition 4.3.1 *Let dg_v denote a left(right) Haar measure on G_v normalized so that*

$$\int_{H_v} dg_v = 1$$

for almost all $v \notin J_\infty$. We know that Haar measure is necessarily finite on compact sets, so we can normalize Haar measure. Then there is a unique left (right) Haar measure dg on G such that for each finite set of indices S containing J_∞, the restriction of dg_s of dg of G_s is precisely the product measure. We will write $dg = \prod_v dg_v$ for this measure.

Proof: Let us choose S as as set containing J_∞ and define product of measure dg_v as dg_S. By the normalization of dg_v and the fact that S is finite, then the compact group $\prod_{v \notin S} H_v$ has finite measure with respect to dg_S . As such, dg_S is a Haar measure on G_S. Hence, the product measure dg_S is a radon measure and, furthermore, is invariant under the componentwise group operation because each of the dg_v is invariant under the group operation. See Chapter 7, Theorem 7.28, in Folland's Real Analysis [12]. Let s be a larger set of indices which implies that $G_s \leq G_T$. Then, by construction, we have that dg_S coincides with the restriction of dg_T to the subgroup G_S. Then G admits a Haar measure, dg, as it is locally compact. Furthermore, the restriction of dg to G_S is also a Haar measure on G_S. As such, we can pick any finite set S of indices containing J_∞,and choose the Haar measure dg of G, such that dg restricts to dg_S. Let $S^{'}$ be a set of indices containing J_∞. Then dg, constructed relative to dg_S, uniquely picks out the product measure on $G_{S \cup S'}$, and hence on $dg_{S'}$.Therefore, dg is independent of the S chosen and is unique.

□

We have now properly defined the existence of Haar measure dg on G in terms of Haar measure dg_v on G_v. We want to establish similar ideas in taking the Fourier transformation functions defined on G. In the following proposition, we define a special function which will play important role in proof of the functional equation and analytic continuation of the Hecke L-function.

Proposition 4.3.2 *(i) Let f be an integrable function on G. Then*

$$\int_G f(g) = lim_S \int_{G_S} f(g_s) dg$$

where the limit is taken over larger and larger S. If f is only assumed to be continuous,then the above identity holds, but then we must accept that the integral may take infinite values.

(ii) Let S_0 denote the finite set of indices containing both J_∞ and the set of indices for which $Vol(H_v, dg_v) \neq 1$. Suppose that for each index v, we are given a continuous and integrable function f_v on G_v, such that $f_v|H_v = 1$ for all v outside some finite set S_1. Then for $g = (g_v) \in G$ we can define the function

$$f(g) = \prod_v f_v(g_v)$$

The function f is well defined and continuous on G. Furthermore, if S is any finite set of indices including S_0 and S_1, then we have

$$\int_{G_s} f(g)dg = \prod_{v \in S} \left(f_v(g_v)dg_v\right)$$

Furthermore, if the RHS of above equation is less than ∞*, then*

$$\int_{G} f(g)dg = \prod_{v} \left(f_v(g_v)dg_v\right)$$

and $f \in L^1(G)$

(iii) Let $\{f_v\}$ *and* f *be as they were in the previous part, but with the added constraint of* f_v *being a characteristic function of* H_v *for all* $v \notin S^1$*. Then* $f \in L^1(G)$ *and in abelian case, the Fourier transform of* f *is given by*

$$\hat{f}(g) = \prod_v \hat{f}_v(g_v)$$

If we additionally assume that $\hat{f}_v \in L^1(\hat{G}_v)$ *for all* v*, then* $\hat{f} \in L^1(\hat{G})$*. Recall that* $B(G)$ *is a set of functions such that* f *is continuous,* $f \in L^1(G)$ *and* $\hat{f} \in L^1(G)$*. This is the set of functions for which the Fourier inversion theorem holds. Therefore, if we assume* $f_v \in B(G_v)$ *for all* v*, and both* $Vol(H_v, dg_v)$ *and* $f_v = 1_{H_v}$ *for all but finitely many* v*, then* $f \in B(G)$*.*

Proof: Refer Chapter 5 in Ramakrishnan and Valenza [24].

□

In the next proposition, we will construct the dual measure $d\chi$ to dg on G that the Fourier inversion theorem holds. The Fourier inverion theorem is a key ingredient in proving both the Poisson summation formula and the Riemann-Roch theorem.

Proposition 4.3.3 *The measure* $d\chi = \prod_v d\chi_v$*, where* $d\chi_v = \hat{dg_v}$*, is dual measure of* $dg = \prod_v dg_v$*. Therefore,*

$$f(g) = \int_{\hat{G}} \hat{f}(\chi)\chi(g)d\chi$$

for all $f \in B(G)$

Proof: From the previous propositions, we can say that $Vol(H_v, dg_v) = 1$ and $d\chi = \prod_v d\chi_v$ is a Haar measure on $\hat{\hat{G}}$. Now we have to check duality for a given product of functions. We already knwo that the Fourier transform for the set of functions such that $f = \prod_v f_v$, where $f_v \in B(G_v)$ and $f_v = 1_{H_v}$. By part (iii) of above proposition, such functions are a part of $B(G)$ and we have that

$$\int_{\hat{G}} \hat{f}(\chi)\chi(g)d\chi = \prod_v \int_{\hat{G}_v} \hat{f}_v(\chi_v)\chi_v(g_v)d\chi_v$$

Since $d\chi_v$ is the dual measure to dg_v, then

$$f_v g_v = \int_{\hat{G}_v} \hat{f}_v(\chi_v)\chi_v(g_v)d\chi_v$$

Therefore,

$$\int_{\hat{G}} \hat{f}(\chi)\chi(g)d\chi = \prod_v f_v(g_v) = f(g)$$

□

4.4 Adeles and Ideles

Let K be a number field. Let K_v be the completion of K at the v^{th} place of K. The restricted direct product of K_v, under addition, with respect to o_v,is called the adele group of K, and is denoted $\mathbb{A}_K$. Here K_v is an abelian locally compact group and o_K is a compact-open subgroup of K_v for all finite places v of K. Every element of K is divisible by finitely many prime ideals, and hence the embedding of K into K_v for all v lies in o_v for all but finitely many places. Therefore, K embeds diagonally into $\mathbb{A}_K$:

$$K \to \mathbb{A}_K$$

$$x \mapsto (x, x, ...)$$

Similarly, we can define the idele group, denoted by $\mathbb{I}_K$. It is a restricted direct product of K_v^*, as the multiplicative group with respect to o_v^*, an open compact subset of K_v^*. As every element of K^* is locally an integer,and hence a unit for all but finitely many places, it diagonally embeds into $\mathbb{I}_K$:

$$K^* \to \mathbb{I}_K$$

$$x \mapsto (x, x, , , , ,)$$

Proposition 4.4.1 *(Approximation Theorem) For every global field K,we have both*

$$\mathbb{A}_K = K + A_\infty \text{ and } K \cap A_\infty = o_K$$

Proof: We know that K embeds diagonally into $\mathbb{A}_K$.We must show that for every element $x \in \mathbb{A}_K$, there exists an element $k \in K$ such that $x - k$ has an absolute value less than or equal to 1 for all finite places; meaning that, $x - k$ is locally an integer for all finite places. Let p_v be the prime ideal of o_K corresponding to v, where v is finite place of K. Let $x = (x_v) \in \mathbb{A}_K$. For all v, there exists a positive integer m_v such that $p_v^{m_v} x_v \in o_K$. Since $x \in \mathbb{A}_K$ is locally not an integer at only a finite number of places, then we may find a rational integer m which we can implicitly diagonally embed into $\mathbb{A}_K$ such that all finite components of mx lie in the ring of integers. We can construct a set of prime ideals where e_j be the power of prime p_j appearing in the unique factorization of the ideal m in o_K.

$$m = p_1^{e_1} p_2^{e_2} p_2^{e_2} ....p_n^{e_n}$$

Now by applying Chinese Remainder Theorem, we can find a $\lambda \in o_K$ such that

$$mx_j = \lambda(modp_j^{e_j'})\ j = \{1, 2, 3....\}$$

where x_j is the $p_j th$ component of x in adeles. Let $k = \lambda m$, then $x - k$ will be integral at $p_1...p_n$. We know that ring of integers of a global field K have finite intersections and $\mathbb{A}_\infty$ consists of all elements of the adeles that are locally an integer at all finite places. So, $K \cap A_\infty = o_K$.

□

Corollary 4.4.2

$$\mathbb{A}_\mathbb{Q} = \mathbb{Q} + \mathbb{A}_\infty = \mathbb{Q} + \left(\mathbb{R} \times \prod_{pprime} \mathbb{Z}_p\right)$$

and $\mathbb{Q} \cap \mathbb{A}_\infty = \mathbb{Z}$

Lemma 4.4.3 *Let E/K be a finite extension and fix a K-basis $\{u_1, u_2...\}$ of E. Then the map*

$$\alpha : \prod_{j=1}^{n} \mathbb{A}_K \to \mathbb{E}$$

$$((x_{v,j})_v)_j \mapsto \sum_j u_j (x_{v,j})_v$$

is an isomorphism of topological groups

Proof: See Ramakrishnan and Valenza [24], Chapter 5, Section 3, Lemma 5-10.
□

Proposition 4.4.4 *K is a discrete, co-compact subgroup of $\mathbb{A}_K$.*

Proposition 4.4.5 *There exists an isomorphism of topological groups*

$$\mathbb{A}_{\mathbb{Q}}/\mathbb{Q} \cong \lim \mathbb{R}/n\mathbb{Z}$$

Proof: See Ramakrishnan and Valenza [24], Chapter 5, Section 3, Proposition 5-12.

□

Proposition 4.4.6 *The group K^* embeds discretely in $\mathbb{I}_K$.*

Proof: We know that $\phi : \mathbb{I}_K \to \mathbb{A}_K^2$, which yields a topological isomorphism of $\mathbb{I}_K$ onto its image under ϕ. We know from Proposition 4.4.4 that K embeds discretely into $\mathbb{A}_K$. Here $K \times K$ embeds discretely to $\mathbb{A}_K \times \mathbb{A}_K$, which implies that $K^* \times K^*$ embeds discretely in $\phi(\mathbb{I}_K)$.

□

Definition 4.4.7 We define the idele-class group to be $\mathbb{I}_K/K^*$ and we denote it by C_K.

Since $\mathbb{A}_K/\mathbb{I}_K$ is compact, one might hope that C_K is also compact. But this is not true, it follows from the existence of a nontrivial absolute value that will be defined shortly. But first we must standardize our absolute value functions:

Definition 4.4.8 Let F be a local field of characteristic zero. We define the normalized absolute value on F as follows:

(i) If $F = \mathbb{R}$, then let $|.|_F$ be the standard absolute value.

(ii) If $F = \mathbb{C}$, then let $|.|_F$ be the square of the standard absolute value.

(iii) If F is non-Archimedean, then let $|.|_F$ be such that $|\pi_F|_F = \frac{1}{q}$ where π_F is uniformizing parameter of F , and q is the order of the residue field $o_F/\pi_F o_F$.

These normalized absolute values satisfy another very important property. Let l/k be finite extension of fields. If one fixes a basis of l over k, then we know that every endomorphism of l as a k-vector space is uniquely representable as a matrix with entries in k. Since l is a field, every element x of l defines an endomorphism ρ_x of l as a k-vector space via multiplication. This formally is called the regular representation. We can define the norm of x by $N_{l/k}(x)$ as the determinant of the matrix representation of ρ_x.

Lemma 4.4.9 *Let l/k be a finite extension of local fields. Then for all $x \in l$, we have*

$$|x|_l = |N_{l/k}(x)|_k$$

Proof: Let k be non-Archimedean with the uniformizing parameter π_k. Let π_l be the uniformizing parameter of l. Let $q_k = [o_k : \pi_k o_k]$ and $q_l = [o_l : \pi_l o_l]$. Every element in l can be written uniquely in the form $u\pi_l^m$ for some $m \in \mathbb{Z}$ and some $uo_l^\times$. Let e be the ramification index of l/k and e is determined by the relation $\pi_k = v\pi_l^e$ for some $v \in o_l^\times$. Let f be the residual degree of l/k. That is, f is determined by the relation $q_1 = q^f$. From the above propositions, we can say that

$$|\pi_k|_l = mod_l(\pi_K) = mod_k(\pi_K)^n = |\pi_K|_k^n = q^{-n}$$

where $n = [l : k]$. With our choice of e and f, we get $n = ef$. Since the uniformizing parameter is only unique up to a unit in the ring of integers, then we can replace π_k with $v^{-1}\pi_k$, so that $\pi_k = \pi_l^e$. As $\pi_l^e \in k$, then

$$N_{l/k}(\pi_l^e) = \pi_k^n$$

and thus we state

$$|N_{l/k}(\pi_l^e)|_k = |\pi_k^n|_k = \frac{1}{q^n} = \frac{1}{q^{ef}}$$

Hence, from the definition, we have that

$$|\pi_l|_l = \frac{1}{q_{[1]}} = \frac{1}{q^f}$$

□

Theorem 4.4.10 *Let K be a number field. Then:*

(i) For every $x \in K^*$ we have $|x_{\mathbb{A}_K}| = 1$. It is usually called Artin's product formula

(ii) The absolute value map $|.|_{\mathbb{A}_K}$ is surjective.

Since $|.|_{\mathbb{A}_K}$ is continuous and surjective map from $\mathbb{I}_K$ to $\mathbb{R}_+^\times$ with $K^* \subset Ker(|.|_{\mathbb{A}_K})$, then the quotient group $C_K = \mathbb{I}_K/K^*$ cannot be compact.

Definition 4.4.11 Let K be a algebraic number field. We define the ideles of norm one to be

$$\mathbb{I}_K^1 := Ker(|.|_{\mathbb{A}_K})$$

As K^* is a subgroup of $\mathbb{I}_K^1$ by the above theorem, we define the norm-one idele-class group to be the quotient group $C_K^1 = \mathbb{I}_K^1/K^*$.

The above theorem implies that the following sequence is short exact:

$$1 \to C_K^1 = \mathbb{I}_K^1/K^* \xrightarrow{inc} C_K = \mathbb{I}_K/K^* \xrightarrow{|.|_{\mathbb{A}_K}} \mathbb{R}^\times \to 1$$

Theorem 4.4.12 *Let K be a number field. The quotient group $C_K^1 = \mathbb{I}_K^1/K^*$ is compact*

Proof: See Ramakrishnan and Valenza [24], Chapter 5, Theorem 5-15.

□

5 Tate's Thesis

In this chapter, we have followed the presentation of Ramakrishnan and Valenza [24], while also referring to Tate [27], Koch [16], Lang [19], and Kudla [18] for some details and ideas.

5.1 Local Quasi-Characters and their Associated Local L-factors

Let F be a local field and let $|.|_F$ be the normalized absolute value, as defined in previous chapter. The unit group $F^\times$ of a local field F is the direct product of $o_F^\times \times V(F)$, where $o_F^\times$ is the subgroup of $F^\times$ of elements with absolute value 1 and

$$V(F) = \{y \in \mathbb{R}_+^\times : y = |x|_F; x \in F^\times\}$$

In the non-Archimedean case, $o_F^\times$ is the group of units in the ring of integers of F, and $V(F) = q^{\mathbb{Z}}$, where q is the order of residue field o_F/po_F for p as the unique prime ideal in F.

Definition 5.1.1 A $\chi \in Hom(F^\times, \mathbb{C}^\times)$ is unramified if it is trivial on the group of units $o_F^\times$ of F.

Proposition 5.1.2 *For every unramified quasi-character χ of $F^\times$ there exists a complex number s such that $\chi(\alpha) = |\alpha|_F^s$ for all $\alpha \in F^\times$*

Proposition 5.1.3 *For every character χ of $F^\times$ has the form*

$$\chi(x) = \tilde{\chi}(\tilde{x})|x|_F^s$$

where $\tilde{\chi}$ is a unitary character of $o_F^\times$, $\tilde{x}$ is the continuous homomorphism of $F^\times$ and $o_F^\times$, and $s \in \mathbb{C}$. The real part of s is uniquely determined by the quasi-character, but the imaginary part of s is not, as $|.|^{i\tau}$ for $\tau \in \mathbb{R}$ is a unitary character. We denote by σ the real part of s and call it the exponent of χ.

Proof: Let χ be a quasi-character and denote by $\tilde{\chi}$ the restriction of χ to $o_F^\times$. Since $o_F^\times$ is compact, $\tilde{\chi}$ is a continuous homomorphism of $o_F^\times$ into $\mathbb{C}^\times$, then $\tilde{\chi}(o_F^\times)$ is a compact subgroup of $\mathbb{C}^\times$ and hence is contained in S^1. Thus, $\tilde{\chi}$ is a character of $o_F^\times$. We can define the continuous homomorphism $x \mapsto \chi(x)\tilde{\chi}(\tilde{x})^{-1}$ and is an unramified quasi-character of $F^\times$. Now using the result from previous proposition, we have that $\chi(x)\tilde{\chi}^{-1}(\tilde{x}) = |x|_F^s$ for some $s \in \mathbb{C}$.

□

Two quasi-characters are called equivalent if their quotient is an unramified quasi- character. This relation certainly is reflexive, transitive, and symmetric, and hence an equivalence relation. Each equivalence class is isomorphic to the space of unramified quasi-characters. We now will describe the space of quasi-characters with the equivalence relation for the three types of local fields.

(i) If $F = \mathbb{R}$, then he space of quasi-characters is a pair of complex-planes.

(ii) If $F = \mathbb{C}$, then the space of quasi-characters is a countable set of complex planes indexed by the integers.

(iii) If F is non-Archimedean, then the space of quasi-characters is a countable set of cylinders

$$\{s \in \mathbb{C} : s \cong s^{'} if s - s^{'} = m\tfrac{2\pi i}{log(q)}, m^{'} \in \mathbb{Z}\}$$

Let us recall the Gamma function, which is given by the integral

$$\tau(z) = \int_0^\infty e^{-z} t^{z-1} dt$$

Definition 5.1.4 Let F be a local field and let $Hom(F^\times, \mathbb{C}^\times)$

(i) If $F = \mathbb{C}$, then let

$$L(\chi_{s,n}) = \tau_{\mathbb{C}}(s + \tfrac{|n|}{2}) = (2\pi)^{-(s+\frac{|n|}{2})}\tau(s + \tfrac{|n|}{2})$$

(ii) If $F = \mathbb{R}$ and $\chi = \tilde{\chi}|.|^s$, then let

$$L(\chi) = \begin{cases} \tau_{\mathbb{R}}(s) = \pi^{s/2}\tau(s/2) & \tilde{\chi} = 1 \\ \tau_{\mathbb{R}}(s+1) & \tilde{\chi} = sgn \end{cases} \tag{1}$$

(iii) If F is non-Archimedean, then let

$$L(\chi) = \begin{cases} (1 - \chi(\pi_F))^{-1} & \chi is unramified \\ 1 & otherwise \end{cases} \tag{2}$$

where π_F is the uniformizing parameter, a generator of the unique maximal ideal, p of F.

We have observed that each equivalence class of quasi-characters is a surface that is isomorphic to the whole complex plane, or a quotient group of the complex plane. Therefore, we can say that $L(\chi)$, for a given local field F , is a function on the domain of quasi-characters of F . In this way, it makes sense to say that $L(\chi)$ is a meromorphic, nonzero, function of $s \in \mathbb{C}$. Thus, on each equivalence class of quasi-characters, $L(\chi)$ is a meromorphic function of $s \in \mathbb{C}$.

Given any quasi-character χ of $F^{\times}$ and a complex number s, the product $\chi|.|_F^s$ is also a character. We define the shifted dual for χ as

$$\tilde{\chi} = \chi^{-1}|.|_F$$

And thus

$$L((\chi|.|_F^s)) = L(\chi^{-1}|.|_F|.|_F) = L(1-s, \chi^{-1})$$

5.2 Local Additive Characters and the Self-Duality of Local Fields

For understanding the self-duality of local fields, we will need to establish the existence of a non-trivial additive character. We will now construct the standard non-trivial additive characters for each of the local fields.

(i) ($F = \mathbb{R}$) Let $\phi(x) = e^{-2\pi ix}$. We have $\phi(x) \neq 1$ if and only if $x \in \mathbb{R} - \mathbb{Z}$ and it is continuous.

(ii) ($F = \mathbb{C}$) Let $\phi(x) = e^{-2\pi i tr_{\mathbb{C}/\mathbb{R}}(x)}$, where $tr_{\mathbb{C}/\mathbb{R}}(x) = x + \overline{x} = R(x)$. We have $\phi(x) \neq 1$ if and only if $R(x) \neq \mathbb{Z}$. It can be verified easily that ϕ is a continuous homomorphism of $\mathbb{C}$ into S^1.

(iii) (F non-Archimedean). First, we will define a non-trivial character on $\mathbb{Q}_p$ for some rational prime p, and then use the trace map, which is additive, to define a character on a finite extension of $\mathbb{Q}_p$.

Theorem 5.2.1 *Let ψ be a fixed nontrivial unitary additive character of the locally compact field F. The existence of such a character was shown above. For each $a \in F$, define $\psi_a : F \to S^1$ by $\psi_a(x) = \psi(ax)$. Then the map $\alpha_\psi : F \to \hat{F}$ given by $a \mapsto \psi_a$ is a topological group isomorphism.*

Proof: We start by proving that α_ψ is a well-defined map and, is an injective group homomorphism of F into its Pontryagin dual $\hat{F}$. We have

$$\psi_a(x+y) = \psi(a(x+y))) = \psi(ax+ay) = \psi(ax)\dot{\psi}(ay) = \psi_a(x)\dot{\psi}_a(y)$$

As ψ is a homomorphism from F to S^1, $|\psi_a(x)| = 1$. Since left multiplication by a is a continuous map from F into itself, it is continuous and unitary character. Similarly, we can define α_ψ, where it will be homomorphism of groups and it will be trivial for $a = 0$. Thus, α_ψ is an injective group homomorphism for F to $\hat{F}$.

Recall that the topology of dual group is compact-open topology and so we can define a neighbourhood of trivial character

$$W(C, V) = \{\chi : \chi(C) \subseteq V\}$$

where C is a compact set of F and V is an open neighborhood of the identity of S^1. We can reformulate the neighborhood base of the trivial character as

$$W(C_m,) = \{\chi : |\chi(x) - 1| < \epsilon; x \in C_m\}$$

where we consider the compact sets of C as $C_m = \{x \in F : |x|_F \leq m\}$ for $m \in \mathbb{R}$ and neighbourhood base of identity of S^1.Now we will simplify the topology of the dual group in order to prove its continuity.

Since ψ is continuous, then for all $\epsilon > 0$ there exists a $\delta > 0$ such that $|\psi(x)-1| < \epsilon$ whenever $|x|_F < \delta$. To show continuity of group homomorphism, we show that for all $W(C_m, \epsilon)$, there exists a neighbourhood $U = \{y \in F : |y|_F < \delta/m\}$ of 0 in F with $\alpha_\psi(U) \subseteq W(C_m, \epsilon)$. For all $y \in U$, we have

$$|\alpha_\psi(y)(x) - 1| = |\psi(xy) - 1| < \epsilon$$

for all $x \in C_m$ because $|yx|_F = |y|_F|x|_F < \delta$. Thus, $\alpha_\psi(U) \subseteq W(C_m, \epsilon)$, which implies it is a continuous injective group homomorphism. Similarly, we can show that its inverse is a continuous map from $\alpha_\psi(F)$ onto F, which in turn proves that α_ψ is a topological isomorphism onto its image.

From the propositions proved in previous chapters, lets recall that $\hat{F}$ is locally compact. Since α_ψ is an open map, which implies $\alpha(F)$ will be open and thus a closed subgroup of the locally compact group $\hat{F}$. Also recall that a closed subgroup of a locally compact group is locally compact in the subspace topology. Thus, α_ψ is surjective.

□

Proposition 5.2.2 *Let G be a locally compact abelian group with Haar measure dx, and let $d\chi$ be the dual measure. it is the measure on the Pontryagin Dual $\hat{G}$ relative to which the Fourier inversion formula holds. Suppose that we have an isomorphism $\alpha : G \to \hat{G}$ of topological groups. Then there exists a unique measure mu such that $\mu = t \cdot dx$ for some $t \in \mathbb{R}_+^\times$ and mu identifies as dual measure under α. It can also be called as self-dual measure on G relative to the isomorphism α*

Proof: Let us recall the formula for Fourier Inversion Theorem of an $f \in L^1(G)$ by

$$\hat{f}(y) = \int_G f(x)\overline{\alpha(y)}(x)dx$$

where $\alpha(y)$ is the unique character in $\hat{G}$ associated to $y \in G$. The Fourier Inversion Theorem asserts the existence of measure such that $\hat{\hat{f}} = f(-y)$ for all $f \in B(G)$ and $y \in G$. As the Haar measures are unique up to a constant, then the Fourier inversion theorem implies that $\hat{\hat{f}} = \frac{1}{t} \cdot f(-y)$ for some constant t defined relative to dx. Thus, if $\mu = t \cdot dx$, then μ identifies with the dual measure of α.

□

5.3 Local Schwartz-Bruhat Functions

Definition 5.3.1 A complex-valued function f on F is smooth if it is C^∞ whenever F is Archimedean, and locally compact otherwise. If F is non-Archimedean, we say f is smooth if $f(x) = f(x_0)$ where x is sufficiently close to x_0. In the Archimedean case, a Schwartz function f on F is a smooth function such that the function, together with all its higher derivatives, vanish at infinity faster than any power of $|x|$. We say f is Schwartz function, if for any non-negative integers N, M

$$sup_{x \in F}(1+|x|)^N |\tfrac{d^M}{dx^M} f(x)| < \infty$$

A Schwartz-Bruhat function is a Schwartz function if F is Archimedean, and is a smooth function with compact support if F is non-Archimedean. Let $S(F)$ denote the space of Schwartz-Bruhat functions.

Examples:

(i) If F is Archimedean, then $f_n(x) = x^n e^{-|x|^2}$ is a Schwartz-Bruhat function for any nonnegative interger n.

(ii) If F is non-Archimedean, then the characteristic functions of compact sets of F are Schwartz-Bruhat. Examples of compact sets of F are p^n for n, a non-negative integer, where p, the unique prime of F .

Proposition 5.3.2 *For every $f \in S(F)$, F is non-Archimedean, there exists n, m, $-m \leq n$, such that $f(x) = 0$ for $x \notin p^{-m}$, and for $x \in p^{-m}$, $f(y) = f(x)$ for all $y \in x + p^n$*

Proof: Let $x \in supp(f)$. Since f is locally constant, then there exists an open neighborhood U_x of x such that $f(U_x) = f(x)$. Moreover, since $\{p^n\}_{n \in N}$ forms a neighbourhood basis for $0 \in F$, then by homogeneity, we may take $U_x = x + p^{n(x)}$ for some $n(x) \in \mathbb{N}$. Then there exists an open cover of $supp(f)$. Since the support of f is compact, then finite number of the U_x cover the support. Thus, there exists $x_1, x_2...x_r \in supp(f)$ such that $supp(f) \subseteq \cup_{i=1}^{r} U_x$. let $n = minn(x_i)$. Then $supp(f) \subseteq \cup_{i=1}^{n}(x+p^n)$. Now as the Heine-Borel theorem holds for a non-Archimedean local field, then $supp(f)$, which is compact, is bounded. Also, every bounded set in F is contained in some p^m.

□

5.4 The Meromorphic Continuation and Functional Equation of the Local Zeta Function

Definition 5.4.1 For $f \in S(F)$ and $\chi \in Hom_{cont}(F^\times, \mathbb{C}^\times)$, we define the local zeta function

$$Z(f,\chi) = \int_{F^\times} f(x)\chi(x)d^*x$$

Here $Z(f,\chi)$ is dependent on the multiplicative measure d^*x. if we fix an additive measure dx and choose $d^*(x) = dx/|x|_F$, then $Z(f,\chi)$ will be dependent on dx.

Recall, $Z(f,\chi)$ is a function on the domain of quasi-characters of F. Since each equivalence class of quasi-characters is a surface that is isomorphic to either the whole complex plane or a quotient group of the complex plane, then we may speak of the analytic continuation from one subset of an equivalence class to a larger subset. In the next theorem, we first will show that $Z(f,\chi)$ is a holomorphic and absolutely convergent function in the domain of quasi-characters of exponent ($\sigma = R(s)$) greater than 1. We will also show the analytic continuation of the zeta function to a function in the domain of quasi-characters of all exponents.

Theorem 5.4.2 *Let $f \in S(F)$, and $\chi = \tilde{\chi}|.|^s$ where $\tilde{\chi}$ is the unitary part of the quasi-character χ. Let $\sigma = R(s)$. Then the following statements hold:*

(i) $Z(f,\chi) = Z(f,\tilde{\chi},s)$ is holomorphic and absolutely convergent if $\sigma > 0$.

(ii) If $0 < \sigma < 1$, then there is a functional equation

$$Z(\hat{f},\dot{\chi}) = \gamma(\chi,\psi,dx)Z(f,\chi)$$

for some $\gamma(\chi,\psi,dx)$, which is independent of f and meromorphic as a function of s.

(iii) There exists a factor $\epsilon(\chi, \psi, dx)$ that lies in $\mathbb{C}^\times$ for all s and satisfies the relation

$$\gamma(\chi, \psi, dx) = \epsilon(\chi, \psi, dx)\frac{L(\dot{\chi})}{L(\chi)}$$

therefore the relation

$$L(\chi)Z(\hat{f}, \dot{\chi}) = \epsilon(\chi, \psi, dx)L(\dot{\chi})Z(f, \chi)$$

illustrates that the poles of $Z(f, \chi)$ are similar to $L(\chi)$, which is independent of f. Thus, $L(\chi) = Z(f_0, \chi)$ for some suitable f_0

Proof: Refer to Chapter 7 in Ramakrishnan and Valenza [24]

□

Definition 5.4.3 For any multiplicative character $\omega : o_F^\times \to S^1$ and an additive character $\lambda : o_F \to S^1$, define to be Gauss Sum to be

$$g(\omega, \psi) = \int_{o_F^\times} \omega(u)\lambda(u)d^*u$$

The generalization of Gauss sums was an important part of Tate's thesis. A Gauss sum will be appear in the epsilon factor for ramified quasi-characters.

Lemma 5.4.4 *Let ω and λ be taken as above with conductors p^n and p^r respectively. Let $c > 0$ be the number such that $d^*dx = cdx$. Then the following statements hold :*

(i) If $r < n$, then $g(\omega, \lambda) = 0$

(ii) If $r = n = 0$, then $|g(\omega, \lambda)|^2 = Vol(o_F^\times, d^*x)^2$

(iii) If $r = n$, then $|g(\omega, \lambda)|^2 = cVol(o_F, dx)Vol(U_r, d^*x)$

(iv) If $r > n$, then $|g(\omega, \lambda)|^2 = cVol(o_F, dx)(Vol(U_r, d^*x) - q^{-1}Vol(U_{r-1}, d^*x))$

Proof: If $r = n = 0$, then $\omega_{o_F^\times} = 1$ and $\lambda|_{o_F^\times} = 1$. Therefore, we have

$$g(\omega, \psi) = \int_{o_F^\times} \omega(u)\lambda(u)d^*u = Vol(o_F^\times, d^*x)$$

And hence $|g(\omega,\lambda)|^2 = Vol(o_F^\times, d^*x)^2$. Let R be a residue system of $o_F^\times/U_r o_F^\times$ in $o_F^\times$. For $a \in R$ and $1 + b\pi_F^r \in U_r$ we have

$$\lambda(a(1+\pi_F^r b)) = \lambda(a)\lambda(a\pi_F^r b) = \lambda(a)$$

because $p^r = \pi_F^r o_F$ is the conductor of λ. Then

$$g(\omega,\lambda) = \sum_{a \in R} \lambda(a)\omega(a) \int_{U_r} \omega(u) d^*u$$

if $r < n$, then there exists an element $u_0 \in U_r$ such that $\omega(u_0) \neq 1$. Using the invariance of multiplicative Haar measure, we obtain

$$\int_{U_r} \omega(u) d^*u = \int_{U_r} \omega(uu_0) d^*u = \omega(u_0) \int_{U_r} \omega(u) d^*u = 0$$

This proves (i). Suppose $r \geq n$. Applying the transformation $x = zy$ and translation invariance of the Haar measure, we obtain

$$g(\omega,\psi) = \int_{o_F^\times} \omega(x)\lambda(x) d^*x \cdot \overline{\int_{o_F^\times} \omega(y)\lambda(y) d^*y} = \int_{o_F^\times} \omega(z) h(z) d^*z$$

where

$$h(z) = \int_{o_F^\times} \lambda(y(z-1)) d^*y$$

For $z - 1 \in p^r$ and $y \in o_F$, we have that $\lambda(y(z-1)) = 1$. On the other hand, if $z - 1 \in \{p^r - p^{r-1}\}$, then there exists a $y_0 \in o_F^\times \subset o_F$ such that $\lambda(y_0(z-1)) = 1$. Thus, we obtain

$$h(z) = \begin{cases} c(1-q^{-1})Vol(o_F, dx) & z-1 \in p^r \\ -cq^{-1}Vol(o_F, dx) & z-1 \in \{p^{r-1} - p^r\} \\ 0 & otherwise \end{cases} \quad (3)$$

Using the above information and applying it to $|g(\omega, \lambda)|^2$, parts (ii) and (iii) now follow at once.

□

With the knowledge of the above lemma, we can do some computations of $Z(f_n, \chi_{s,n})$ for $n > 0$. As the conductor of ψ is p^{-d}, then the conductor of ψ_{π_k} is p^{-d-k}. before the lemma it was determined that

$$Z(f_n, \chi_{s,n}) = \sum_{k=-d-n} q^{-ks} g(\tilde{\chi}, \psi_{\pi_F^k})$$

where $\tilde{\chi}$ has conductor p^n. From part (i) of the above lemma, we have $g(\tilde{\chi}, \psi_{\pi_F^k}) = 0$ for all $k > -d-n$. Therefore,

$$Z(f_n, \chi_{s,n}) = q^{(d+n)s} g(\tilde{\chi}, \psi_{\pi_F^{-d-n}})$$

Now as both $\tilde{\chi}$ and $\psi_{\pi_F^{-d-n}}$ have conductor p^n, then from part (ii) and (iii) of the above lemma, we can say that $g(\tilde{\chi}, \psi_{\pi_F^{-d-n}}) \neq 0$. Thus, $Z(f_n, \chi_{s,n})$ is an exponential function with neither zeroes or poles. Lets recall that for $n > 0$, $L(\chi_{s,n})$ is 1 because $\chi_{s,n}$ is not ramified. Thus,

$$Z(f_n, \chi_{s,n}) = q^{(d+n)s} g(\tilde{\chi}, \psi_{\pi_F^{-d-n}}) L(\chi_{s,n})$$

Now let us find out the Fourier transform of our function f

Lemma 5.4.5 *For $n = 0$, we have $\hat{f}_0(y) = Vol(p^{-d}, dx) 1_{o_F}(y)$, where $1_{o_F}(y)$ is the characteristic function of o_F. For $n > 0$, we have $\hat{f}_n(y) = Vol(p^{-d-n}, dx) 1_{p^n-1}(y)$, where $1_{p^n-1}(y)$ is the characteristic function of $p^n - 1$.*

Proof: By definition,

$$\hat{f}_n(y) = \int_F f_n(x) \psi_{xy} dx = \int_{p^{-d-n}} \psi(x)\psi(xy) = \int_{p^{-d-n}} \psi(x(y+1)) dx$$

First, let $n = 0$. The conductor of ψ is p^{-d}. For $y \notin o_F$, we have $\psi(x(y+1))$ is non-trivial for some $x \in p^{-d}$, hence $\hat{f}(y) = 0$ by orthogonality of characters. On the other hand, $y \in o_F$, $\psi(x(y+1)) = 1$ for all $x \in p^{-d}$, hence $\hat{f}(y) = Vol(p^{-d}, dx)$. Similarly, we repeat the steps for $n > 0$, and we can establish that $\hat{f}(y) = Vol(p^{-d-n}, dx)$

□

5.5 Local Epsilon Factor and Root Number

As we have observed the local epsilon factor in Theorem 5.4.2, here we will state a Proposition for the dependence of epsilon factor $\epsilon(\chi, \psi, dx)$ on both the additive character ψ and Haar measure dx for any $\chi \in Hom(F^\times, \mathbb{C}^\times)$

Proposition 5.5.1

(i) For every real number t,

$$\epsilon(\chi, \psi, t \cdot dx) = t \cdot \epsilon(\chi, \psi, dx)$$

(ii) Let $a \in F^\times$ and let $\psi(a)$ be the character defined as $\psi_a(x) = \psi(ax)$. Then

$$\epsilon(\chi, \psi_a, dx) = \chi(a)|a|_F^{-1}\epsilon(\chi, \psi, dx)$$

(iii) Let F be a non-Archimedean field with unique prime ideal p, and let p^n and p^{-d} be the conductors of χ and ψ, respectively. Then for every unramified character v of $F^\times$ we have

$$\epsilon(\chi v, \psi, dx) = v(\pi^{d+n})\epsilon(\chi, \psi, dx),$$

where π is the uniformizing parameter for o_F

(iv)

$$\epsilon(\dot{\chi}, \psi, dx) = \frac{\chi(-1)}{\epsilon(\chi, \psi, dx)}$$

(v)

$$\epsilon(\overline{\chi}, \psi, dx) = \chi(-1)\overline{\epsilon(\chi, \psi, dx)}$$

Proof: (i) Since the Fourier transform of a self-dual local field is dependent on the Haar measure, dx chosen for F and the additive character, ψ, chosen, then we will denote the Fourier transform of a function $f \in S(F)$ by $(\hat{f}, \psi, dx)$. By definition we have that

$$(\hat{f}, \psi, tdx) = \int_F f(x)\psi(xy)tdx = t\int_F f(x)\psi(xy)dx = t(\hat{f}, \psi, dx)$$

Although, $Z(f,\chi)$ is dependent on d^*x, therefore on dx we set $d^*x = dx/|x|_F$, the ratio $Z(\hat{f},\dot{\chi})/Z(f,\chi)$ is independent of the measure chosen, whether we specify the multiplicative measure independent of dx or not. Therefore, we have

$$\frac{Z(\hat{f},\psi,dx),\dot{\chi},tdx}{Z(f,\chi,tdx)} = t\frac{Z(\hat{f},\psi,dx),\dot{\chi},dx}{Z(f,\chi,tdx)}$$

Hence, we have

$$\epsilon(\chi,\psi,t\cdot dx) = t\cdot\epsilon(\chi,\psi,dx)$$

(ii) With the same notation in part (i), the results follows with similar steps

(iii) Since v is unramified, then there exists an $s^{'}\in\mathbb{C}$ such that $v = |.|_F^{s^{'}}$. We also know that, $\chi = |.|_F^s\tilde{\chi}$ for some $s\in\mathbb{C}$ and unitary $\tilde{\chi}$, the restriction of χ to o_F, with conductor p^n. The conductor of χv is the same as the conductor of χ since v is unramified. As in the local computations, we write $\chi v = \chi_{s+s',n}$. If $n = 0$, then

$$\epsilon(\chi_{s+s',0},\psi,dx) = q^{-d(s+s^{'}-1)}Vol(o_F,dx)$$

Now for $n > 0$

$$\epsilon(\chi_{s+s',n},\psi,dx) = q^{-d(s+s^{'}-1)}q^{-n(s+s^{'})}Vol(o_F,dx)\sum_{x\in U/U_m}\overline{\tilde{\chi}}\psi_{\pi_F^{-d-n}}(x) = v(\pi_F^{d+n})\epsilon(\chi_{s,n},\psi,dx)$$

For (iv) and (v), the result follows by applying the part (ii) of Theorem 5.4.2 and translation invariance of Haar measure.

□

Definition 5.5.2 Let F be a local field with standard non-trivial character ψ and self-dual measure dx. For a unitary character $\tilde{\chi}$ of $F^\times$, one defines the root number $W(\tilde{\chi})$ by

$$W(\tilde{\chi}) = \epsilon(\tilde{\chi}|.|_F^{1/2},\psi,dx)$$

Proposition 5.5.3 $|W(\tilde{\chi})| = 1$

Proof: The result follows from the parts (iv) and (v) of the above proposition.

□

5.6 Adelic Schwartz-Bruhat Functions and the Riemann-Roch Theorem

One of the most important and useful results of abelian harmonic analysis is the Poisson summation formula, which relates the averages of a function over a lattice to its Fourier transform. The Poisson summation formula will help us to establish the global functional equation.

Definition 5.6.1 Let K be a global field. Let v be a place of K and K_v be the completion of K with respect to v. Define

$$S(\mathbb{A}_K) = \otimes_v^{'} S(K_v) = \{f = \otimes f_v : f_v \in S(K_v) \forall v; f_v = 1_{o_v}\}$$

where 1_{ov} is the characteristic function of o_v. A function $f \in S(\mathbb{A}_K)$ is called an Adelic Schwartz-Bruhat function.

According to the Proposition in adeles chapter, we can write

$$f(x) = \prod_v f_v(x_v)$$

for all $x = (x_v) \in \mathbb{A}_K$.

Proposition 5.6.2 *For each place v of K, let ψ_v be the standard unitary character on K_v. Then the restriction of ψ_v to o_v is trivial for almost all v. Hence*

$$\psi_K\big(\prod_v x_v\big) = \prod_v \psi_v(x_v) \text{ for } x = (x_v) \in \mathbb{A}_K$$

is a well-defined non-trivial character on the adeles. Furthermore, $\psi(\alpha) = 1$ for $\alpha \in K$

Proof: Recall that the conductor of ψ_v is the inverse different of K_v. Since the inverse different is trivial for all but finitely many places v, then $\psi_v|_{o_v} = 1$ for all but finitely many places v, and hence $\prod_v \psi_v$ is a well defined character on $\mathbb{A}_K$. Firstly, us first restrict ourselves to $K = \mathbb{Q}$ in order to show that ψ is trivial on the embedding of $K = \mathbb{Q}$ into $\mathbb{A}_\mathbb{Q}$. Recall that if $\alpha \in \mathbb{Q}$, then there is a unique expansion of the form

$$\alpha = \sum_p \frac{a_p}{p^{v_p}} + b$$

where a_p, v_p, $b \in \mathbb{Z}$ and $a_p = 0$ for all but finitely many primes. Applying this unique representation, we get

$$\psi_{\mathbb{Q}}(\alpha) = \prod_p \psi_p(\alpha) = \psi_\infty(a) \prod \psi_p(\frac{a_p}{p^{v_p}}) = e^{-2\pi i a} \prod_p e^{\frac{2\pi a_p}{p^{v_p}}} = e^{-2\pi b} = 1$$

We also know that

$$\sum_{v|p} tr_{K_v/\mathbb{Q}_p}(.) = tr_{K/\mathbb{Q}}(.)$$

See Neukrich's text, Algebraic Number Theory

Then, for a finite extension K of $\mathbb{Q}$, we have

$$\psi_K(\alpha) = \prod_p \prod_{v|p} \psi_p(tr_{K_v/\mathbb{Q}_p(\alpha)}) = \prod_v \psi_p(tr_{K/\mathbb{Q}}(\alpha)) = 1$$

□

Definition 5.6.3 Let f be a complex-valued function on $\mathbb{A}_K$ such that both $\hat{f}$ and $\hat{\hat{f}}$ are normally convergent; both are absolutely and uniformly convergent on compact sets. Then we say that f is admissible.

Lemma 5.6.4 *All $f \in S(\mathbb{A}_K)$ are admissible.*

Proof: Let $f \in S(\mathbb{A}_K)$. Let C be compact set of $\mathbb{A}_K$. We know that compact sets of local field K_v are of the form $p_v^{n_v}$ where p is the unique prime ideal of K_v and $n_v \in \mathbb{Z}$.

$$\prod_{V \in S_\omega} C_v \times \prod_{V \in S} p_v^{n_v} \times \prod_{V \notin S \cup S_\omega} o_v$$

where S is the finite set of finite places such that $f|_{o_v} \neq 1$ and S_ω is set of infinite places. Since the characteristic functions of $p_v^{m_v}$ generate $S(K_v)$, so we can assume that f_v for all $v \in S$ are characteristic functions. Suppose that $f(\gamma + z) \neq 0$ for some $z \in C$ and $\gamma \in K$. Since we assumed that for all $v \in S$, f_v is a characteristic function of $p_v^{m_v}$, then the vth components of $\gamma + z \in \mathbb{A}_K$, $v \in S$ are necessarily in p^{m_v} for all $v \in S$. Consequently, $\gamma \in p^{m_v} - z_v \subset p_v^{k_v}$ for all $v \in S$. Therefore,

$$|\hat{f}(z)| = |\sum_{\gamma \in K} f(\gamma + z)| \leq \sum_{\gamma \in K} \prod_v |f_v(\gamma + z_v)| = \sum_{\gamma \in I} |f_\omega(\gamma + z_\omega)|$$

where

$$f_\omega = \prod_{v \in S_\omega} f_v \in S(\prod v \in S_\omega K_v)$$

In the previous chapter, we have show that K is discrete subgroup of $\mathbb{A}_K$, which means the fractional ideal I will be a discrete subgroup of $\prod_{v \in S_\omega} K_v$, where K_v

is isomorphic to $\mathbb{R}$ or $\mathbb{C}$. We have also shown that Fourier transform maps $S(\mathbb{A}_K)$ to $S(\mathbb{A}_K)$. Thus, $\hat{f}$ is normally convergent.

□

Now we shall proceed to prove the Poisson Summation Formula.

Theorem 5.6.5 *Let $f \in S(\mathbb{A}_K)$. Then $\tilde{f} = \hat{\tilde{f}}$; that is*

$$\sum_{\gamma \in K} f(\gamma + x) = \sum_{\gamma \in K} \hat{f}(\gamma + x)$$

for all $x \in \mathbb{A}_L$

Proof: If $\phi \in \mathbb{A}_K$ is a K-invariant, then ϕ induces a function on $\mathbb{A}_K/K$. So we can apply Fourier transform $\phi : \mathbb{A}_K/K \to \mathbb{C}$ on K,as it is dual group of $\mathbb{A}_K/K$. For all $z \in K$

$$\hat{\phi}(z) = \int_{\mathbb{A}_K/K} \phi(t)\psi_K(tz)\overline{dt}$$

where $\overline{dt}$ is the quotient Haar measure on compact group $\mathbb{A}_K/K$ induced by dt on $\mathbb{A}_K$. The quotient measure $\overline{dt}$ is characterised by the relation

$$\int_{\mathbb{A}_K/K} \tilde{f}(t)\overline{dt} = \int_{\mathbb{A}_K/K} (\sum_{\gamma \in K} f(\gamma + t))\overline{dt} = \int_{\mathbb{A}_K} f(t)dt$$

for all continuous and admissible functions f on $\mathbb{A}_K$. In order to proceed, we will establish two lemmas.

Lemma 5.6.6 *For every $f \in S(\mathbb{A}_K)$, we have*

$$\hat{f}|_K = \hat{\tilde{f}}|_K$$

Proof: Let us fix $z \in K$. Using the fact $\psi_K|_K = 1$, then by definition, we have

$$\hat{\tilde{f}}(z) = \int_{\mathbb{A}_K/K} \tilde{f}(t)\psi_K(tz)\overline{dt} = \int_{\mathbb{A}_K/K} (\sum_{\gamma \in K} \tilde{f}(\gamma + t))\psi_K(tz)\overline{dt}$$

$$= \int_{\mathbb{A}_K/K} (\sum_{\gamma \in K} \tilde{f}(\gamma + t)\psi_K(\gamma + t)z))\overline{dt} = \int_{\mathbb{A}_K} f(t)\psi_K(tz)dt = \hat{f}(z)$$

□

Lemma 5.6.7 *For every $f \in S(\mathbb{A}_K)$, and $x \in K$, we have*

$$\tilde{f}(x) = \sum_{\gamma \in K} \hat{\hat{f}}(\gamma)\overline{\psi_K}(\gamma x)$$

Proof: As $\hat{f}|_K = \hat{\tilde{f}}|_K$, then

$$|\sum_{\gamma \in K} \hat{\tilde{f}}(\gamma)\overline{\psi_K}(\gamma x)| = |\sum_{\gamma \in K} \hat{f}(y)\overline{\psi_K}(\gamma x)| \leq \sum_{\gamma \in K} |\hat{f}(\gamma)|$$

where ψ_K is unitary. Therefore, the expression on the right-hand side of the lemma is normally convergent, since $f \in S(\mathbb{A}_K)$ is admissible. Also,

$$\sum_{\gamma \in K} \hat{\tilde{f}}(\gamma)$$

is convergent for the same reason. Since K is discrete, then

$$\sum_{\gamma \in K} \hat{\tilde{f}}(\gamma)\overline{\psi_K}(\gamma x)$$

is the Fourier transform of $\hat{\tilde{f}}$ at $-x$, and from Fourier Inversion Theorem, we know that $\hat{\hat{\tilde{f}}}(-x) = \tilde{f}(x)$. Thus, the lemma is proved.

□

Let us return to the proof of the Poisson summation formula. Applying the second lemma with $x = 0$ and then the first lemma, we obtain

$$\tilde{f}(0) = \sum_{\gamma \in K} \hat{\tilde{f}}(\gamma)\overline{\psi_K}(0) = \sum_{\gamma \in K} \hat{\tilde{f}}(\gamma) = \sum_{\gamma \in K} \hat{f}(\gamma)$$

Since $\tilde{f}(0) = \sum_{\gamma \in K} f(\gamma)$, then

$$\sum_{\gamma \in K} f(\gamma) = \sum_{\gamma \in K} \hat{f}(\gamma)$$

□

We now will proceed with the number field analogue of the geometric Riemann-Roch Theorem.

Theorem 5.6.8 *(Riemann-Roch) Let $x \in \mathbb{I}_K$. Let $f \in S(\mathbb{A}_K)$. Then*

$$\sum_{\gamma \in K} f(\gamma x) = \frac{1}{|x|_{\mathbb{A}_K}} \sum_{\gamma \in K} \hat{f}(\gamma x^{-1})$$

Proof: Let us fix an $x \in \mathbb{I}_K$. Since, f is admissible, then we define $f_x(y) = f(xy)$, is in $S(\mathbb{A}_K)$, is admissible. So, the sum on the left is normally convergent. The Poisson summation formula applied to f_x yields

$$\sum_{\gamma \in K} f_x(\gamma) = \sum_{\gamma \in K} \hat{f}_x(\gamma)$$

Upon computing the Fourier transform of f_x, we have

$$\hat{f}_x(\gamma) = \int_{\mathbb{A}_K} f(xy)\psi_K(y\gamma) = \frac{1}{|x|_{\mathbb{A}_K}} \int_{\mathbb{A}_K} f(y)\psi_K(yx^{-1}\gamma)dy = \frac{1}{|x|_{\mathbb{A}_K}} \hat{f}(\gamma x^{-1})$$

This completes the proof.

□

5.7 Idele-Class Characters

Proposition 5.7.1 *Every idele-class character χ has the factorization $\chi = \tilde{\chi}|.|^s$ where $\tilde{\chi}$ is a unitary character.*

Proof: Let $\chi \in Hom(\mathbb{I}_K/K^*, \mathbb{C}^\times)$. Let v_∞ be the infinite place of K. Suppose we have a subgroup $V(\mathbb{I}_\mathbb{K}) = \{(t_{v_\infty}, 1, 1, ...) : t_{v_\infty \in \mathbb{R}_+^\times}\}$ of $\mathbb{I}_K$. Thus, we have a map $|.|_{\mathbb{A}_K} = V(\mathbb{I}_K) \to \mathbb{R}_+^\times$ is an isomorphism. Since we uniquely can write any idele in the form $x = |x|_{\mathbb{A}_K} \cdot y$ where $y \in \mathbb{I}_K^1$, then the map $\phi : V(\mathbb{I}_K) \times \mathbb{I}_K^1 \to \mathbb{I}_K$, defined by $(\alpha, \beta) \mapsto \alpha\beta$ is an isomorphism. Moreover, we have the short exact sequence

$$1 \to C_K^1 = \mathbb{I}_K^1/K^* \to C_K = \mathbb{I}_\mathbb{K}/K^* \to V(\mathbb{I}_K) = \mathbb{R}_+^\times \to 1$$

Recall that C_K^1 is compact. Since the quasi-character is continuous, it will form a compact subgroup and will be contained in S^1. Therefore, $\chi|_{\mathbb{I}_K^1/K^*} = \tilde{\chi}$ is a unitary character on $\mathbb{I}_K^1/K^*$. Now, $\tilde{\chi}^{-1}\chi$, by definition, is trivial on $\mathbb{I}_K^1/K^*$.

Therefore, an arbitrary quasi-character on C_K is of the form $\alpha \mapsto \tilde{\chi}(\tilde{a})|\alpha|^s$, where $\tilde{\alpha}$ is characterized by the relation $\alpha = \tilde{\alpha}\beta$ for some unique $\beta \in C_K^1$.

□

An idele-class character, χ, is called unramified if $\chi|_{\mathbb{I}_1} = 1$. We say that two idele-class characters are equivalent if their quotient is unramified. Each equivalence class is of the form

$$\{\tilde{\chi}|.|^s : s \in \mathbb{C}\}$$

for some fixed unitary character $\tilde{\chi}$

5.8 The Meromorphic Continuation and Functional Equation of the Global Zeta Function

Let K be a number field and let ψ_K the standard adelic character. Let dx_v be the self-dual additive measure with respect to ψ_v. We set

$$d^*x_v = \tfrac{q_v}{q_v - 1} \cdot \tfrac{dx_v}{|x_v|_v}$$

as the Haar measure of the multiplicative group of the completion of K with respect to finite places, v, of K.

Definition 5.8.1 Let $\chi \in Hom(\mathbb{I}_K/K^*, \mathbb{C}^\times)$. For $f \in S(\mathbb{A}_K)$, define global zeta function by

$$Z(f, \chi) = \int_{\mathbb{I}_K} f(x)\chi(x)d^*x$$

Here f will be necessarily continuous on $\mathbb{I}_K$ as its restricted direct product topology is stronger than the subspace topology induced by $\mathbb{A}_K$.

Recall that the local zeta function was a function on the domain of quasi-characters of a local field F ,similarly, $Z(f, \chi)$ is a function on the domain of idele-class characters of a given number field K. In the following theorem, we first will prove that $Z(f, \chi)$ is absolutely and uniformly convergent on the domain of idele-class characters of exponent greater than 1. Then we will prove that in the equivalence class of unramified characters, $Z(f, \chi)$ can be meromorphically continued to the whole s-plane with two simple-poles at $s = 0$ and $s = 1$. Thus, on all other equivalence classes, $Z(f, \chi)$ can be analytically continued to the whole s-plane.

Theorem 5.8.2 *For all idele-class characters $\chi = \tilde{\chi}|.|^s$ and $f \in S(\mathbb{A}_K)$, the global zeta function $Z(f,\chi)$ is normally convergent in $\sigma = R(s) > 1$. Furthermore, $Z(f,\chi)$ extends to a meromorphic function of s and satisfies the unctional equation*

$$Z(f,\chi) = Z(\hat{f},\dot{\chi})$$

The continuation is entire in all classes of idele-class characters except for the class of unramified characters, which is given by the set

$$\{\chi \in Hom(\mathbb{I}_K/K^*, \mathbb{C}^\times) : \tilde{\chi} = |.|^{-i\tau}; \tau \in \mathbb{R}\}$$

For a given class representative $\chi = |.|^{s-i\tau}$, $Z(f,|.|^{s-i\tau})$ has simple poles at $s = i\tau$ and $s = i + i\tau$, with corresponding residues given by

$$-Vol(C_K^1)f(0) \text{ and } Vol(C_K^1)f(0)\hat{f}(0)$$

respectively. The volume of C_K^1 is taken with respect to the quotient measure on C_K defined by both d^*x and the counting measure on K^*.

Proof: Since $f \in S(\mathbb{A}_K)$, then f_v is the characteristic function of o_v for all but finitely many finite places v of K. Let S be the finite set of finite places for which $f_v \in S(K_v)$ is not a characteristic function of o_v. For all finite places v of K, let p_v be the unique prime of K_v and let π_v be a uniformizing parameter of p_v. We may take f_v for $v \in S$ to be a characteristic function of $p_v^{m_v} = \pi_v^{m_v} o_v$ by linearity and translation invariance of the Haar measure. Let S_ω be the set of infinite places of K. As such, the product

$$\prod_v c_v \int_{K_v - \{0\}} |f_v(x_v)||x_v|_v^{\sigma-1} dx_v$$

where $c_v = q_v/q_v - 1$ for finite places and $c_v = 1$ for infinite places. Recall from part (i) of Theorem 5.4.2, for Archimedean fields, we showed that

$$\int_{K_v - \{0\}} |f_v(x_v)||x_v|_v^{\sigma-1} dx_v$$

is finite for $\sigma > 0$. Since the number of infinite places is finite, then the product of the Archimedean integrals is equal to some positive real M. For non-

Archimedean fields, we can use the result from product of integrals to determine the convergence of the global zeta function, which is given by the infinite product

$$\prod_{V \notin S \cup S_\omega} \frac{1}{1-q_v^{-\sigma}}$$

An infinite product of complex numbers is said to converge if the sequence of the partial products has nonzero limits. If we fix the principal branch of logarithm, then $\prod_{n=1}^{\infty} a_n$ converges if and only if the series $\sum_{n=1} log a_n$ converges , where log denotes the principal branch of the logarithm. See Alfhors, Complex Analysis Chapter V 2.2 ([1]). A product is called absolutely convergent if the series converges absolutely. Therefore, in order to determine the region of convergence of the product

$$\sum_{V \notin S \cup S_\omega} log\left(\frac{1}{1-q_v^{-\sigma}}\right) = \sum_{V \notin S \cup S_\omega} \sum_{m=1}^{\infty} \frac{q_v^{-m\sigma}}{m}$$

Hence, by using p-test,we can say that the above function converges. Now since, $Z(f, \chi)$ is convergent, by using the proposition from ideles and adeles chapter, we find that $Z(f, \tilde{\chi}, s)$ is normally convergent in $\sigma = R(s) > 1$. In order to show that $Z(f, \chi)$ is holomorphic for $\sigma > 1$, we need to just exchange the order of the derivative d/ds and the integral.

Now moving to the second part of the theorem. if we fix K at an infinite place, tehn $\mathbb{I}_K \cong \mathbb{R}_+^\times \times \mathbb{I}_K^1$. Now if $\sigma > 1$, then applying the Fubini's Theorem with both $f \in S(\mathbb{A}_K)$ and $\sigma \geq 1$, we obtain

$$Z(f, \chi) = \int_{\mathbb{I}_K} f(x)\chi(x) d^*x = \iint_{\mathbb{R}_+^\times \times \mathbb{I}_K^1} f(tx)\chi(tx) \frac{dt}{t} d^*x$$

where the product tx takes place at the fixed infinite component of x. We define

$$Z_t(f, \chi) = \int_{\mathbb{I}_K} f(tx)\chi(tx) d^*x$$

We will now apply Riemann-Roch theorem to establish a functional equation for above

Proposition 5.8.3 *The function $Z_t(f, \chi)$ satisfies the relation*

$$Z_t(f,\chi) = Z_{t^{-1}}(\hat{f},\dot{\chi}) + \hat{f}(0)\int_{C_K^1} \dot{\chi}(x/t)d^*x - f(0)\int_{C_K^1} \chi(tx)d^*x$$

Proof: By definition, $C_K^1 = \mathbb{I}_K^1/K^*$. Since K^* is discrete in $\mathbb{I}_K^1$, then the Haar measure on K^* is the counting measure. Then

$$Z_t(f,\chi) = \int_{C_K^1} (\sum_{a\in K^*} f(atx)\chi(atx))d^*x = \int_{C_K^1} (\sum_{a\in K^*} f(atx))\chi(tx)d^*x$$

since $\chi|_{K^*} = 1$ by hypothesis. To apply the Riemann-Roch theorem, we need to sum over K, not K^*. So, we get

$$Z_t(f,\chi) + f(0)\int_{C_K^1} \chi(tx)d^*x = \int_{C_K^1} (\sum_{a\in K^*} f(atx))\chi(tx)d^*x$$

Applying the Riemann-Roch theorem to the sum on the right-hand side and then using the change of variable x to x^{-1}, we get

$$\begin{aligned}\int_{C_K^1} (\sum_{a\in K^*} f(atx))\chi(tx)d^*x &= \int_{C_K^1} (\sum_{a\in K^*} \hat{f}(at^{-1}x^{-1}))\frac{\chi(tx)}{|tx|_{\mathbb{A}_K}}d^*x \\ &= \int_{C_K^1} (\sum_{a\in K^*} \hat{f}(at^{-1}x))|t^{-1}x|_{\mathbb{A}_K}\chi(tx^{-1})d^*x \quad (4) \\ &= Z_{t^{-1}}(\hat{f},\dot{\chi}) + f(0)\int_{C_K^1} \dot{\chi}(tx)d^*x\end{aligned}$$

since $\dot{\chi} = \chi^{-1}|.|$. This completes the proof of the Proposition. □

Now returning back to our theorem. We will split the zeta function in the following way:

$$Z(f,\chi) = \int_0^1 Z_t(f,\chi)\frac{1}{t}dt + \int_1^\infty Z_t(f,\chi)\frac{1}{t}dt$$

We see that

$$\int_1^\infty Z_t(f,\chi)\frac{1}{t}dt = \int_{x\in\mathbb{I}_K} f(x)\chi(x)d^*x$$

The integral on the right-hand side is normally convergent for $\sigma > 1$. Therefore, the integral is normally convergent for all $s \in \mathbb{C}$. We now will use the functional equation for $Z_t(f,\chi)$ to investigate the integral from 0 to 1. By applying the change of variable $t \to t^{-1}$ like before , we get

$$\int_0^1 Z_{t^{-1}}(\hat{f},\dot{\chi})\frac{1}{t}dt = \int_1^\infty Z_t(\hat{f},\dot{\chi})\frac{1}{t}dt$$

which is convergent for all σ by the argument above. Now we have to analyze

$$R(f,\chi) := \int_0^1 \hat{f}(0)\dot{\chi}^{-1}\int_{C_K^1}\dot{\chi}(x)d^*x\frac{1}{t}dt - \int_0^1 f(0)\chi(t)\int_{C_K^1}\chi(x)d^*x\frac{1}{t}dt$$

There are two cases to consider : (i) if χ is nontrivial in $\mathbb{I}_K$ and (ii) if $\chi = \tilde{\chi}|.|^s$. In either case, we have

$$Z(f,\chi) = \int_1^\infty Z_t(\hat{f},\dot{\chi})\frac{1}{t}dt + \int_1^\infty Z_t(f,\chi)\frac{1}{t}dt + R(f,\chi)$$

We have that $\hat{\hat{f}}(x) = f(-x)$, since dx is self-dual relative to ψ_K on $\mathbb{A}_K$. In addition, $\ddot{\chi} = \chi$ by definition. Applying these two facts, we obtain

$$Z(\hat{f},\dot{\chi}) = \int_1^\infty Z_t(\hat{f},\dot{\chi})\frac{1}{t}dt + \int_1^\infty Z_t(\hat{\hat{f}},\ddot{\chi})\frac{1}{t}dt + R(\hat{f},\dot{\chi})$$

We observe that $R(f,\chi) = R(\hat{f},\dot{\chi})$. Furthermore, χ is idele class character, hence trivial on K^*, so $\chi(-tx) = \chi(tx)$. Thus, we have

$$Z(f,\chi) = Z(\hat{f},\chi).$$

□

5.9 Hecke L-Functions

Let $\chi \in Hom(\mathbb{I}_K/K^*, \mathbb{C}^\times)$ (an idele-class character), for a number field K. We have seen from the propositions in previous chapter that χ can be written as $\tilde{\chi}|.|^s_{\mathbb{A}_K}$ where $\tilde{\chi}$ is the unitary character and $s \in \mathbb{C}$. We can define a local character

$$\chi_v : K_v^* \to \mathbb{C}^\times$$

$$t \mapsto \chi\{1, 1, ..1, t, 1, , , 1\}$$

where t is the v^{th} component. Then $\chi_v = \prod_v \chi_v(y)$.

Definition 5.9.1 *We define the global L-function of χ in terms of its local versions by the product expansion*

$$L(\chi) = \prod_v L(\chi_v)$$

whenever this is convergent.

Lemma 5.9.2 *$L(\chi)$ is absolutely convergent, nonzero, and holomorphic whenever the exponent $\sigma = R(s)$ of χ is greater than* 1

Proof: $L(\chi)$ is nonzero because $L(\chi_v)$ is nonzero for all quasi-characters χ_v. We can write $\chi = \tilde{\chi}|.|^s$ with $\sigma = Rs$. By definition we have that $L(\chi_v) = 1$ if v is a finite place and χ_v is ramified. Since $\chi_v|_{o_v} = 1$ for all but finitely many finite places, then χ_v is unramified for almost all v. In addition, there are only a finite number of non-Archimedean places, v; $L(\chi, v)$ is holomorphic for all $R(s) > 0$ since they come from gamma functions. In order to show that the product is convergent for $\sigma > 1$, then we must show that the logarithm of the product converges for $\sigma > 1$. This result was obtained in the previous chapter. Thus, the result follows.

□

Definition 5.9.3 Let $\chi \in Hom(\mathbb{I}_K/K^*, \mathbb{C}^\times)$ (an idele-class character). For a complex s, define the complex Hecke L-function $L(s, \chi)$ by

$$L(s, \chi) = L(\chi|.|^s)$$

Let us consider the trivial idele-class character $\chi = 1$. Note that $\chi = 1$ belongs to the class of unramified idele-class characters. Then

$$L(s, 1_f) = \prod_{v finite} \frac{1}{1 - |\pi_v|^s} = \prod_{v finite} \frac{1}{1 - N(p_v)^{-s}}$$

where p_v is the unique prime associated to the completion of K at v and N is the absolute norm. Thus, $N(p_v) = [o_K : p_v o_K] = [o_v : p_v o_v] = q_v$.

For an arbitrary number field K, $L(s, 1_f)$ is called the Dedekind zeta function of K and is denoted $\zeta_K(s)$.

For $K = \mathbb{Q}$, then, for $R(s) > 1$, we have that

$$L(s, 1_f) = \prod_p \frac{1}{1 - p^{-s}} = \sum_{n \geq 1} \frac{1}{n^s}$$

is the Riemann zeta function.

Theorem 5.9.4 *Let χ be a unitary idele class character. Then $L(s, \chi)$, which is a priori defined and holomorphic in $R(s) > 1$, admits a meromorphic continuation to the whole s-plane, and satisfies the functional equation*

$$L(1 - s, \overline{\tilde{\chi}}) = \epsilon(s, \tilde{\chi}) L(s, \tilde{\chi})$$

where

$$\epsilon(s, \tilde{\chi}) = \textstyle\prod_v \epsilon(\tilde{\chi}_v |.|^s, \psi_v, dx_v) \in \mathbb{C}^\times$$

for some choice of self-dual pair. The global epsilon factor is, in fact, independent of the this pair.

Proof: Refer Chapter 7, Theorem 7-19 in Ramakrishnan and Valenza [24]

□

5.10 The Volume of C_K^1 and the Regulator

Let K be a number field. Our main goal is to compute the $Vol(C_K^1)$. Recall that the volume of $Vol(C_K^1$ is taken with respect to the quotient measure on C_K, defined by d^*x and the counting measure on K^*.

Let us define $|.|_v$ for all completions of K at v. For a finite set S of places of K, let us define the set of S-ideles of K by

$$\mathbb{I}_{K,S} = \{x = (x_v) \in \mathbb{I}_K : |x_v|_v = 1; \forall v \notin S\}$$

If $S = \emptyset$, then $\mathbb{I}_K \subseteq \mathbb{I}_K^1$. However, even if S is not the empty-set, then we define the norm-one version of the S-ideles by

$$\mathbb{I}_{K,S}^1 := \mathbb{I}_K^1 \cap \mathbb{I}_{K,S}$$

We will follow Ramakrishnan and Valenza [24], Chapter 7, Section 4, and will find the volume in three steps.

Step One : Let us assume that S is nonempty. We know that K^* is a subgroup of $\mathbb{I}_K^1$, but not necessarily of $\mathbb{I}_{K,S}^1$. Consider the following projection map

$$\rho : C_K^1 = \mathbb{I}_K^1/K^* \to (\mathbb{I}_K^1/K^*)/(\mathbb{I}_{K,S}^1 \cdot K^*/K^*)$$

Clearly, $Ker\rho : \mathbb{I}_{K,S}^1 \cdot K^*/K^*$. Thus, we can obtain a short exact sequence of abelian groups.

$$1 \to \mathbb{I}_K^1/K^* \cap \mathbb{I}_{K,S}^1 \to C_K^1 \to C_{K,S} \to 1$$

Let h_S denote the order of $C_{K,S}$. Therefore,

$$Vol(C_K^1) = h_S Vol(\mathbb{I}_K^1/K^* \cap \mathbb{I}_{K,S}^1)$$

We now are reduced to computing the volume of the second factor.

Step Two : Take $S = S_\infty$ as the set of Archimedean places of K. Let r_1 be the number of real places. Let r_2 be the number of complex places (one half of the number of conjugate embeddings). Let $|.|$ denote the usual complex absolute value, which restricts to the usual real absolute value. Define

$$\lambda : \mathbb{I}_{K,S_\infty} \to \mathbb{R}^{r_1+r_2}$$

$$(x_v) \mapsto log|(x_v)|_{v \in S_\infty}$$

Then we have

$$\lambda((x_v \cdot y_v)_v) = (log(|x_v \cdot y_v|))_{v\in S_\infty} = (log(|x_v|) + log(|y_v|))_{v\in S_\infty} = \lambda((x_v)) + \lambda((y_v))$$

Therefore, λ is a homomorphism of groups. Let H denote the hyperplane in $\mathbb{R}^{r_1+r_2}$ given by

$$H := \{t = (t_v) \in \mathbb{R}^{r_1+r_2} : \sum_{vreal} t_v + 2\sum_{vcomplex} t_v = 0\}$$

This construction is analogous to the Minkowski lattice theory used to prove Dirichlet's unit theorem. See Neukirch [23], Chapter 1, Sections 4, 5, and 7, for a proof of the Dirichlet's unit theorem.

Lemma 5.10.1 *The logarithm map has the following properties:*

(i) $Im(\lambda) = H$

(ii) $Ker(\lambda) = \mathbb{I}^1_{K,\emptyset}(= \mathbb{I}_{K,\emptyset})$

□

Let us define $R_S := K \cap \mathbb{A}_{K,S}$, the ring of S-integers of K, where

$$\mathbb{A}_{K,S} = \{x \in \mathbb{A}_K : x_v \in o_v, \forall v \notin S\}$$

Then $R_\infty = K \cap \mathbb{A}_{K,S_\infty}$ consists of the elements that are in o_v for all finite places v. Since $o_v = \bigcap_{vfinite} o_v$, then $R_{S_\infty} = o_K$. Therefore,

$$R_S^\times = K^* \cap \mathbb{I}_{K,S}$$

which implies $o_K^\times = R_{S_\infty}^\times = K^* \cap \mathbb{I}^1_{K,S}$

Definition 5.10.2 We will call the restriction of λ to $K^* \cap \mathbb{I}^1_{K,S_\infty} = o_K^\times$ the regulator map and denote as reg(x). The above lemma tells us that

$$Ker(reg) = K^* \cap \mathbb{I}^1_{K,\emptyset}$$

Step Three : By definition, $\mathbb{I}_{K,\emptyset}$ admits the product decomposition

$$\prod_{vreal} o_v \times \prod_{vcomplex} o_v \times \prod_{vfinite} o_v$$

Let us construct the product Haar measure $d^\times x$ on $\mathbb{I}^1_{K,\emptyset}$ as follows:

(i) For v real, we let $d^\times x_v$ be a counting measure on $o_v^\times = \{\pm 1\}$

(ii) For v complex, we let $d^\times x_v$ be the Lebesgue measure on $o_v^\times = S1$.

(iii) For v finite, we let $d^{\times}x_v = d^*x_v$, the normalized measure on K_v^*, such that $Vol(o_v^{\times}, d^*x_v) = N(D_v)^{-1/2}$, where D_v is different of K_v.

Then,

$$Vol(o_v) = \begin{cases} 2 & forvreal \\ 2\pi & forvcomplex \\ N(D)^{-1/2} & forvfinite \end{cases} \tag{5}$$

we get relative to this measure,

$$Vol(\mathbb{I}_{K,\emptyset}, d^{\times}x) = 2^{r_1}(2\pi)^{r_2}|d_K|^{1/2}$$

Let K be a number field. h_k is the class number of K and where R_K is the regulator of K. Thus, from the above equations, we can say that

$$Vol(C_K^1) = \tfrac{2^{r_1}(2\pi)^{r_2}h_K R_K}{w_K\sqrt{|d_K|}}$$

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